ON THE STUDY OF TORUS EQUIVARIANT CR SECTIONS

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ABSTRACT. In this thesis, we study the growth of dimension for the space of torus equivariant CR sections, and get a torus equivariant Siu–Demailly–Grauert–Riemenschneider type criterion on certain CR manifolds. As a corollary, we obtain a criterion that when a holomorphic line bundle is torus equivariantly big.

CONTENTS

1. INTRODUCTION

Finding holomorphic object has always been a main theme in the complex geometry. Let M be a compact complex manifold, since all holomorphic functions on M are constant functions by Liouville's theorem, people study holomorphic sections of a holomorphic line bundle L over M instead, and the behavior of the growth of the dimension for the space of holomorphic sections $H^0(M, L^k) := \{u \in \mathscr{C}^\infty(M, L^k) : \bar{\partial} u = 0\}$ when $k \to \infty$ turns out to be a core issue. The first concerning result shall be Siegel's lemma (c.f. Ma–Marinescu [10, Lemma 2.2.6]), which states that without any positivity assumption for L, there is

$$\dim_{\mathbb{C}} H^0(M, L^k) \lesssim k^n \text{ for all } k \geq 1. \tag{1.1}$$

In 1950's, Kodaira found that if L is positive, then L is big, namely

$$\dim_{\mathbb{C}} H^0(M, L^k) = O(k^n), \tag{1.2}$$

where $n := \dim_{\mathbb{C}} M$. This result can be derived by the combination of vanishing property with the index theorem. On one hand, by the Kodaira–Serre vanishing theorem (c.f. Ma–Marinescu [10, Theorem 1.5.6]), if L is positive, then the higher Dolbeault cohomology of $\left(\mathscr{C}^\infty(M, T^{*0,\bullet}M \otimes L^k), \bar\partial\right)$ vanishes, namely

$$H^q(M, L^k) = 0 \text{ for all } q \geq 1,\ k \gg 1. \tag{1.3}$$

On the other hand, the Riemann–Roch–Hirzebruch theorem suggests that

$$\sum_{q=0}^{n}(-1)^q \dim_{\mathbb{C}} H^q(M, L^k) = \int_M Td(T^{1,0}M)ch(L^k) = \frac{k^n}{n!}\int_M \left(\frac{iR^L}{2\pi}\right)^n + o(k^n) = O(k^n). \tag{1.4}$$

when L is positive. From (1.3) and (1.4), we get Kodaira's (1.2), and this is important because people can hence produce many holomorphic sections, and it is exactly the first step toward the celebrated Kodaira's embedding theorem, which states that a holomorphic line bundle L over a compact complex manifold is positive if and only if L is ample.

In 1970's, Grauert and Riemenschneider tried to generalize the result from Kodaira. Roughly speaking, they quested when a compact complex manifold M is bimeromorphic to a projective one, that is M is Moishezon. It's a known characterization that a manifold M is Moishezon if and only if it carries a big line bundle L, in other words, (1.2) holds (Ma–Marinescu [10], Theorem 2.2.15). In 1983 and 1984, Siu [11] and Demailly [6] find different criteria for the bigness of a semi–positive line bundle, respectively; namely, let M be a compact connected complex manifold of complex dimension n, and (L, h^L) be a Hermitian line bundle over M, then M is Moishezon if one of the following conditions is verified:

(S) (Siu's condition)

$$iR^L \text{ is semi-positive and positive at a point over } M,$$

(D) (Demailly's condition)

$$\int_{M(\leq 1)} \left(\frac{iR^L}{2\pi}\right)^n > 0.$$

Here R^L is the canonical curvature induced by h^L and $M(\leq q) := \cup_{j=1}^{q} M(j)$, where

$$M(q) := \{x \in X : iR^L_x \text{ is non-degenerate with exactly } q \text{ negative eigenvalues}\}.$$

(Note that (S)$\Rightarrow$(D), since in this case $M(1) = \emptyset$, $M(0) \neq \emptyset$, $iR^L \geq 0$ and $iR^L > 0$ on a ball). The condition (D) is a direct corollary of the influential holomorphic Morse inequality, which was first appeared in Demailly [6]. Demailly was inspired by Wittens analytic proof of classical Morse inequality; the role of Morse function is played by the Hermitian metric of L, and the Hessian of the Morse function is replaced by curvature R^L instead. With the study of spectral behavior of the Kodaira Laplacians $\square_k$ on L^k for k large by the method of semi-classical and heat kernel, Demailly successfully established

$$\sum_{j=0}^{q}(-1)^{q-j} \dim_{\mathbb{C}} H^j(M, L^k) \leq \frac{k^n}{n!}\int_{M(\leq q)} (-1)^q \left(\frac{iR^L}{2\pi}\right)^n + o(k^n). \tag{1.5}$$

Combine the case $q = 1$ in (1.5) and the Siegel's lemma (1.1), we can find that L is big when the condition (D) holds . We pause here a while to explain (1.5) more, which leads to the generalization of (1.3) and (1.4). In fact, by some linear algebraic argument, when $q = n$, (1.5) gives the asymptotic Riemann–Roch theorem

$$\sum_{j=0}^{n}(-1)^j \dim_{\mathbb{C}} H^j(M, L^k) = \frac{k^n}{n!}\int_{M(\leq n)}(-1)^q \left(\frac{iR^L}{2\pi}\right)^n + o(k^n).$$

Also, for all $j = 0, \cdots, n$, we can deduce that

$$\dim_{\mathbb{C}} H^j(M, L^k) \leq \frac{k^n}{n!}\int_{M(q)}\left(\frac{iR^L}{2\pi}\right)^n + o(k^n), \tag{1.6}$$

which gives the asymptotic vanishing property

$$\dim_{\mathbb{C}} H^j(M, L^k) = o(k^n) \text{ for all } j \geq 1,\ k \text{ large}$$

when the condition (S) holds.

In some recent progress, the growth order of the equivariant holomorphic sections of equivariant line bundles plays an important role in geometric quantization as well as equivariant complex algebraic geometry. The classical method of Siu [11] and Demailly [6] can not be applied directly to the equivariant setting. In this thesis, for the torus equivariant case, we can reduce the problem to certain CR manifold with an extra torus action. Let M be a compact Hermitian manifold of complex dimension n endowed with a holomorphic torus action T^d, (L, h^L) be a holomorphic line bundle over M with a smooth T^d-invariant Hermitian metric, and T_j be the fundamental vector fields induced by T^d in j-th direction. For $(p_1, \cdots, p_d) \in \mathbb{Z}^d$, let

$$H^0_{p_1,\cdots,p_d}(M, L^k) := \{u \in \mathscr{C}^\infty(M, L^k) : \bar{\partial}u = 0,\ -iT_j u = p_j u \text{ for all } j = 1, \cdots, d\}$$

be the space of T^d-equivariant holomorphic sections. We want to ask what $(p_1, \cdots, p_d) \in \mathbb{Z}^d$ makes

$$\dim_{\mathbb{C}} H^0_{kp_1,\cdots,kp_d}(M, L^k) = O(k^n).$$

This is not so clear even when some positivity conditions for the torus invariant curvature on L holds. The obstruction mainly comes from the torus action may even not be locally free. To overcome this issue, triggered by Hendrick–Hsiao–Li [8], we consider the circle bundle

$$X := \{v \in L^* : \|v\|^2_{h^{L^*}} = 1\}$$

which is a also CR manifold of real dimension $2n + 1$ with a naturally fiberwise circle action. Since this action is CR, transversal, we can take a natural CR, transversal $T^{d+1} = T^d \times S^1$ action on X. By the isomorphism

$$H^0_{p_1,\cdots,p_d}(M, L^k) \cong H^0_{b,p_1,\cdots,p_d,k}(X) := \{u \in \mathscr{C}^\infty(X) : \bar{\partial}_b u = 0;\ \forall j = 1, \cdots, d,\ -iT_j u = p_j u\}$$

where $\bar{\partial}_b$ is the tangential Cauchy–Riemann operator with respect to the natural Reeb's vector field induced by CR, transversal T^{d+1} action, we then turn the problem to the study of torus equivariant CR sections. The main point is that the induce $\mathbb{R}$-action by T^{d+1} is locally free, and it is the semi-classical limit for S^1-action. So we can approximate our object with the known case of circle actions.

Following the framework in Hendrick–Hsiao–Li [8], we consider a real $2n+1$ dimensional compact CR manifold X with a CR transversal torus action T^d. Namely, for each fundamental vector field T_j,

$$\left[T_j, C^\infty(X, T^{1,0}X)\right] \subset C^\infty(X, T^{1,0}X)$$

and there exists $(\mu_1, \cdots, \mu_d) \in \mathbb{R}^d \setminus (0, \cdots 0)$ such that

$$T_x^{1,0}X \bigoplus T_x^{0,1}X \bigoplus \mathbb{C}(\sum_{j=1}^d \mu_j T_j)(x) = \mathbb{C}T_xX \text{ for all } x \in X$$

where $T^{1,0}X$ is the abstract CR structure of X. In this case,

$$T_j\bar{\partial}_b = \bar{\partial}_b T_j \text{ for all } j = 1, \cdots, d.$$

So for all $(p_1, \cdots, p_d) \in \mathbb{Z}^d$, we can consider the $\bar{\partial}_b$ subcomplex $\left(\Omega^{(0,\bullet)}_{p_1,\cdots,p_d}(X), \bar{\partial}_{b,p_1,\cdots,p_d}\right)$, where the q-th Fourier component is given by

$$\Omega^{(0,q)}_{p_1,\cdots,p_d}(X) := \{u \in \Omega^{(0,q)}(X) : -iT_ju = p_ju \text{ for all } j = 1, \cdots, d\}$$

and the q-th cohomology group $H^q_{b,p_1,\cdots,p_d}(X)$.

Let $\alpha := \sum_{j=1}^d \mu_j p_j$, $T_0 := \sum_{j=1}^d \mu_j T_j$, and $(\cdot|\cdot)$ be the T_0-rigid L^2 inner product on $\Omega^{(0,q)}(X)$ such that $(-iT_0u|v) = (u| - iT_0v)$ for all $u, v \in \Omega^{(0,q)}(X)$. Let $L^2_{(0,q)}(X)$ be the completion of $\Omega^{(0,q)}(X)$ with respect to $(\cdot|\cdot)$. We may assume $\{\mu_j\}_{j=1}^d$ is linearly independent over $\mathbb{Q}$, and consequently for the self-adjoint operator $-iT_0$, Spec $(-iT_0) \subset \mathbb{R}$ only consists of eigenvalues; in fact,

$$\beta \in \text{Spec } (-iT_0) \iff \beta = \sum_{j=1}^d \mu_j p_j \text{ for some } (p_1, \cdots, p_d) \in \mathbb{Z}^d$$

and

$$L^2_{(0,q),p_1,\cdots,p_d}(X) = \{u \in L^2_{(0,q)}(X) : -iT_0u = \alpha u\}.$$

We also have

$$H^q_{b,p_1,\cdots,p_d}(X) \cong \ker \Box^{(q)}_{b,\alpha} := \{u \in \ker \Box^{(q)}_b : -iT_0u = \alpha u\} \text{ is a finite dimensional subspace of } \Omega^{(0,q)}(X)$$

where $\Box^{(q)}_b := \bar{\partial}^*_b\bar{\partial}_b + \bar{\partial}_b\bar{\partial}^*_b$ is the Kohn Laplacian determined by T_0. Although $\Box^{(q)}_b$ may not be elliptic or hypoelliptic, the $\Box^{(q)}_b - T_0^2$ is a second order self-adjoint elliptic differential operator.

The main idea, which was suggested to me by Professor Chin-Yu Hsiao, is to approximate the $\mathbb{R}$-action induced from T_0 by a suitable S^1-action, which we now explain. For $(p_1, \cdots, p_d) \in \mathbb{Z}^d$ and $\alpha := \sum_{j=1}^d \mu_j p_j \in \text{Spec } (-iT_0)$, we choose a sequence of rational numbers $\{\mu_{k,j}\}_{k=1}^\infty$ converging to μ_j for each j. Then $\hat{T}_k := \sum_{j=1}^d \mu_{k,j} T_j \to T_0$ and $\alpha_k := \sum_{j=1}^d \mu_{k,j} p_j \to \alpha$ as $k \to \infty$. Put

$$\mathscr{K}^q_{b,\alpha_k} := \{u \in \Omega^{(0,q)}(X) : \Box^{(q)}_b u = 0, -i\hat{T}_ku = \alpha_k u\}.$$

A priori we have $\ker \Box^{(q)}_{b,\alpha} \subset \mathscr{K}^q_{b,\alpha_k}$. In this thesis, we proved that there exists lattice points $(p_1, \cdots, p_d)$ such that $\ker \Box^{(q)}_{b,\alpha} = \mathscr{K}^q_{b,\alpha_k}$. Furthermore, using the results in [4,9], we obtain:

Theorem 1.1 (=Theorem 4.3)**.** *Let X be a compact CR manifold endowed with a transversal, CR torus action on X. Assume X is torus invariantly pseudoconvex and torus invariantly strongly pseudoconvex at a point. For a fixed $(p_1, \cdots, p_d) \in \mathbb{Z}^d$ such that $H^0_{b,p_1,\cdots,p_d}(X) \neq \{0\}$, assume*

$$\lambda_j p_j > 0 \text{ for all } j = 1, \cdots, d$$

and suppose that there exists a constant $C > 0$ such that

$$\inf \left\{ |m^2\alpha^2 - \beta^2| : \beta \in \operatorname{Spec}(-iT_0),\ \beta \neq m\alpha,\ \ker \Box^{(0)}_{b,\beta} \neq \{0\},\ m \in \mathbb{N} \right\} = C > 0$$

(where $\alpha := \sum_{j=1}^{d} \lambda_j p_j$, $\{\lambda_j\}_{j=1}^{d}$ are the transversal data linearly independent over $\mathbb{Q}$), then

$$\dim_{\mathbb{C}} H^0_{b,mp_1,\cdots,mp_d}(X) = \dim H^0_{m\alpha}(X) = O(m^n).$$

Corollary 1.1 (=Corollary 4.1)**.** *Let M be a compact complex manifold of $\dim_{\mathbb{C}} M = n$ with a holomorphic torus action T^d, and (L, h^L) be a holomorphic line bundle over M with a torus invariant smooth hermitian metric. Take any real numbers $\{\mu_j\}_{j=1}^{d}$ linearly independent over $\mathbb{Q}$. If the canonical curvature R^L induced by h^L satisfies $R^L \geq 0$ and $R^L_z > 0$ for some $z \in M$, and suppose that for the given lattice point $(p_1, \cdots, p_d) \in \mathbb{Z}^d$ satisfies*

$$\mu_j p_j > 0 \text{ for all } j = 1, \cdots, d,$$

and a spectral gap such that for all $m \in \mathbb{N}$, and all $(\hat{p}_1, \cdots, \hat{p}_{d+1}) \neq (mp_1, \cdots, mp_d, m)$ with

$$\ker \Box^{(0)}_{\hat{p}_{d+1}} \Big|_{\mathscr{C}^{\infty}_{\hat{p}_1,\cdots,\hat{p}_d}(M, L^{\hat{p}_{d+1}})} \neq \{0\},$$

there is

$$\inf \left| m^2 \left((\sum_{j=1}^{d} \mu_j p_j)^2 + 1 \right) - \left(\sum_{j=1}^{d+1} \mu_j \hat{p}_j \right)^2 \right| > 0.$$

Then for such $(p_1, \cdots, p_d) \in \mathbb{Z}^d$, L is torus equivariantly big, that is

$$\dim_{\mathbb{C}} H^0_{mp_1,\cdots,mp_d}(M, L^m) = O(m^n).$$

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2. Preliminaries

We begin from some basic CR geometry, and recall the results for S^1-action already known in Hsiao–Li [9] and Cheng–Hsiao–Tsai [4].

2.1. **Basic CR geometry.** Let X be a smooth manifold of $\dim_{\mathbb{R}} X = 2n+1$, $n \geq 1$, we say X is a CR manifold if there is a CR structure, denoted by $T^{1,0}X$, such that

(1) $T^{1,0}X$ is a subbundle of $\mathbb{C}TX$ with $\dim_{\mathbb{C}} T^{1,0}_p X = n$ for any $p \in X$.
(2) $T^{1,0}_p X \cap T^{0,1}_p X = \{0\}$ for any $p \in X$, where $T^{0,1}_p X := \overline{T^{1,0}_p X}$.
(3) For $V_1, V_2 \in \mathscr{C}^\infty(X, T^{1,0}X)$, then $[V_1, V_2] \in \mathscr{C}^\infty(X, T^{1,0}X)$, where $[\cdot,\cdot]$ stands for the Lie bracket.

Note that we can always take a non-vanishing global vector field T such that

$$T^{1,0}X \oplus T^{0,1}X \oplus \mathbb{C}T = \mathbb{C}TX.$$

Denote $\langle \cdot, \cdot \rangle$ the paring by duality, and let ω_0 be the globally defined non-vanishing 1-form satisfying

$$\langle \omega_0, T^{1,0}X \oplus T^{0,1}X \rangle = 0 \text{ and } \langle \omega_0, T \rangle = -1.$$

Then the Levi form is defined by

$$\mathcal{L}_x(\tilde{u}, \bar{\tilde{v}}) := \frac{1}{2i} \left\langle \omega_0(x), \left[\tilde{u}, \bar{\tilde{v}}\right](x) \right\rangle,$$

where $\tilde{u}$ and $\tilde{v} \in \mathscr{C}^\infty(X, T^{1,0}X)$, and by Cartan's formula we can also express it as

$$\mathcal{L}_x(\tilde{u}, \bar{\tilde{v}}) = \frac{-1}{2i} \langle d\omega_0(x), u(x) \wedge \bar{v}(x) \rangle$$

i.e.

$$\mathcal{L}_x := \frac{-1}{2i} d\omega_0(x) \Big|_{T^{1,0}X}.$$

Given a Hermitian metric $\langle \cdot | \cdot \rangle$ on $\mathbb{C}TX$, it induces a Hermitian metric on $\mathbb{C}T^*X$, and hence on $\Lambda^r \mathbb{C}T^*X$ by

$$\langle u_1 \wedge \cdots \wedge u_r | v_1 \wedge \cdots v_r \rangle = \det\left(\langle u_j | u_k \rangle_{j,k=1}^r \right).$$

Define $T^{*1,0}X := \left(T^{0,1} \bigoplus \mathbb{C}T\right)^\perp \subset \mathbb{C}T^*X$, and $T^{*0,1}X := \overline{T^{*1,0}X}$. Let the orthogonal projection

$$\pi^{(0,q)} : \Lambda^q \mathbb{C}T^*X \to T^{*(0,q)}X := \Lambda^q(T^{*0,1}X)$$

with respect to this Hermitian metric, then the tangential Cauchy–Riemann operator is defined by

$$\bar{\partial}_b := \pi^{(0,q+1)} \circ d : \mathscr{C}^\infty(X, T^{*0,q}X) \to \mathscr{C}^\infty(X, T^{*0,q+1}X).$$

By Cartan's formula, we can check that

$$\bar{\partial}_b^2 = 0.$$

Also, consider the formal adjoint $\bar{\partial}_b^*$ with respect to the L^2 inner product

$$(f|g) := \int_X \langle f|g \rangle dV_X,$$

where locally

$$dV_X(x) = \sqrt{det \left\langle \frac{\partial}{\partial x_j} | \frac{\partial}{\partial x_k} \right\rangle} dx_1 \wedge \cdots \wedge dx_{2n+1}.$$

Denote $\Omega^{(0,q)}(X) := \mathscr{C}^\infty(X, T^{*0,q}X)$, then the Kohn Laplacian is then defined by

$$\Box_b^{(q)} := \bar{\partial}_b^* \bar{\partial}_b + \bar{\partial}_b \bar{\partial}_b^* : \Omega^{(0,q)}(X) \to \Omega^{(0,q)}(X).$$

Different frm the Kodaira Laplacian $\bar{\partial}^*\bar{\partial}+\bar{\partial}\bar{\partial}^*$ in the case of complex geometry, the $\square_b^{(q)}$ is not elliptic; locally, if we denote $e_1(x),\cdots,e_n(x)$ to be an orthonormal frame of $T_x^{*(0,1)}X$ and $L_1,\cdots,L_n(x)$ be the dual frame of $T_x^{0,1}X$, then we have

$$\square_b^{(q)}=\sum_{j=1}^n L_j^*L_j+\sum_{j,k=1}^n(e_j\wedge e_k^{\wedge,*})[L_j,L_k^*]+\text{lower odeder terms}$$

and

$$\sigma_{\square_b^{(q)}}(x,\xi)=\sum_{j=1}^n|\sigma_{L_j}(x,\xi)|^2.$$

In particular, there is $\sigma_{\square_b^{(q)}}(x,\omega_0)=0$. Moreover, it may even not be hypoelliptic unless the so called $Y(q)$ condition holds (cf. Chen–Shaw [3]).

2.2. **The S^1-equivariant weak CR Morse inequality.** Let X be a compact connected CR manifold endowed with a CR, transversal S^1-action, i.e. the Reeb vector field T induced by S^1-action satisfy

(1) $\left[T,\mathscr{C}^\infty(X,T^{1,0}X)\right]\subset\mathscr{C}^\infty(X,T^{1,0}X)$, and
(2) $T(x)\oplus T_x^{1,0}X\oplus T_x^{0,1}X=\mathbb{C}T_xX$ for all $x\in X$

respectively. Here, T is given by

$$Tu(x):=\left.\frac{\partial}{\partial\theta}\right|_{\theta=0}u(e^{i\theta}\circ x)\text{ for all }u\in\mathscr{C}^\infty(X),\ x\in X.$$

Such global vector field T matters because it commutes with the operator $\bar{\partial}_b$, i.e.

$$T\bar{\partial}_b=\bar{\partial}_bT.$$

So we can we can consider

$$\Omega_m^{(0,q)}(X):=\{u\in\Omega^{(0,q)}(X):Tu=imu\}$$

and the q-th Kohn–Rossi cohomology

$$H_{b,m}^qX:=\frac{\ker\left(\bar{\partial}_{b,m}:\Omega_m^{(0,q)}(X)\to\Omega_m^{(0,q+1)}(X)\right)}{\operatorname{Im}\left(\bar{\partial}_{b,m}:\Omega_m^{(0,q-1)}(X)\to\Omega_m^{(0,q)}(X)\right)}$$

on the $\bar{\partial}_b$ subcomplex

$$\bar{\partial}_{b,m}:\cdots\to\Omega_m^{(0,q-1)}(X)\to\Omega_m^{(0,q)}(X)\to\Omega_m^{(0,q+1)}(X)\to\cdots.$$

Choose a T-rigid Hermitian metric $\langle\cdot|\cdot\rangle$ on $\mathbb{C}TX$, and construct the L^2 inner product $(\cdot|\cdot)$ accordingly, then we also have $T\bar{\partial}_b^*=\bar{\partial}_b^*T$ with respect to $(\cdot|\cdot)$. Hence $\bar{\partial}_b^*|_{\Omega_m^{(0,q)}X}=\bar{\partial}_{b,m}^*$. Put $\square_{b,m}^{(q)}:=\square_b^{(q)}\Big|_{\Omega_m^{(0,q)}(X)}$ accordingly, and by taking $L^2_{(0,q)}(X)$ to be the completion of $\Omega^{(0,q)}(X)$ with respect to the T-rigid L^2 inner product $(\cdot|\cdot)$ induced by $\langle\cdot|\cdot\rangle$, in fact there is the Hodge theorem

$$H_{b,m}^q(X)\cong\mathcal{H}_{b,m}^q(X):=\{u\in\operatorname{Dom}\square_b^{(q)}:\square_b^{(q)}u=0\text{ and }Tu=imu\}.$$

To study the asymptotic bounds for the dimension of S^1 equivariant CR sections, one way is to introduce the Szegö kernel

$$\Pi_m^{(q)}(x) := \sum_{j=1}^{d_m} |f_j(x)|^2 := \sum_{j=1}^{d_m} \langle f_j(x)|f_j(x)\rangle$$

where $\{f_j\}_{j=1}^{d_m}$ is an orthonormal basis for the finite dimensional space $\mathcal{H}^q_{b,m}(X)$. The above definition is in fact independent of the choice of basis; furthermore,

$$\int_X \Pi^{(q)}_{b,m}(x)dV_X(x) = \dim_{\mathbb{C}} \mathcal{H}^q_{b,m}(X) = \dim_{\mathbb{C}} H^q_{b,m}(X).$$

For every $k \in \mathbb{N}$, take

$$X_k := \{x \in X : \forall \theta \in [0, \frac{2\pi}{k}), e^{i\theta} \circ x \neq x \text{ and } e^{i\frac{2\pi}{k}} \circ x = x\}$$

and we define the regular set by

$$X_{\text{reg}} := \{x \in X : \forall \theta \in [0, 2\pi), e^{i\theta} \circ x \neq x\} := X_1.$$

From now on, we all assume $X_{\text{reg}} \neq \emptyset$. In fact, we can also check that X_{reg} is an open dense subset of X, and $X \setminus X_{\text{reg}}$ has measure zero. Also, we collect the information about the positivity of Levi form by

$$X(q) := \{x \in X : \text{the Levi form } \mathcal{L}_x \text{ is non-degenerate and has exactly } q \text{ negative eigenvalues}\}.$$

We are now ready to state one of the main results established by Hsiao–Li in [9]. After careful calculation of the relation of Kodaira Laplcain and Kohn–Rossi Laplacian, semiclassical approximation of Kodaira Laplcain and the Bergman kernel on $\mathbb{C}^n$, the local asymptotic behavior of Szegö kernel can be summarized as:

Theorem 2.1 (Local S^1-equivaraint weak CR Morse inequality)**.** *Assume the same X, then*

(1) $\forall x \in X$, $\sup\{m^{-n}\Pi_m^{(q)}(x) : m \in \mathbb{N},\ x \in X\} < \infty$.
(2) $\forall k \in \mathbb{N}$, $x \in X_k \neq \emptyset$, *then* $\forall q = 0, 1, \cdots, n$

$$\limsup_{m\to\infty} m^{-n}\Pi_m^{(q)}(x) \leq \frac{k^n}{2\pi^{n+1}} |\det L_x| 1_{X(q)}(x).$$

By compactness of X, we can integrate the Szegö kernel and have

$$\begin{aligned}
\limsup_{m\to\infty} m^{-n}\dim_{\mathbb{C}} H^q_{b,m}(X) &= \limsup_{m\to\infty} m^{-n}\dim_{\mathbb{C}} \mathcal{H}^q_{b,m}(X)\\
&= \limsup_{m\to\infty}\int_X m^{-n}\Pi^{(q)}_m(x)dV_X(x)\\
&= \limsup_{m\to\infty}\int_{X_{\mathrm{reg}}} m^{-n}\Pi^{(q)}_m(x)dV_X(x)\\
&\le \int_{X_{\mathrm{reg}}}\limsup_{m\to\infty} m^{-n}\Pi^{(q)}_m(x)dV_X(x)\\
&\quad\text{(Fatou's lemma is guaranteed by the first part of \textbf{Theorem 2.1})}\\
&\le \frac{1}{2\pi^{n+1}}\int_{X_{\mathrm{reg}}} |\det\mathcal{L}_x| 1_{X(q)}(x)dV_X(x)\\
&\quad\text{(take $k=1$ in the second part of \textbf{Theorem 2.1})}\\
&= \frac{1}{2\pi^{n+1}}\int_{X(q)} |\det\mathcal{L}_x| dV_X(x),
\end{aligned}$$

which implies:

Theorem 2.2 (S^1-equivaraint weak CR Morse inequality). *Assume the same X, then as $m\to\infty$*

$$\dim_{\mathbb{C}} H^{(q)}_{b,m}(X) \le \frac{m^n}{2\pi^{n+1}}\int_{X(q)} |\det\mathcal{L}_x| 1_{X(q)} dV_X(x) + o(m^n).$$

Given a holomorphic line bundle with a smooth hermitian metric over a compact Hermitian manifold, the circle bundle plays the role revealing the information for the positivity of the line bundle, so we can apply the result on CR manifold presented earlier to rebuild the well-known theorem:

Corollary 2.1 (Demailly's weak holomorphic Morse inequality). *Let M be a compact Hermitian manifold with $\dim_{\mathbb{C}} M = n$, (L, h^L) be a holomorphic line bundle over M with a smooth hermitian metric. Then $\forall q = 0,\cdots,n$, as $k\to\infty$,*

$$\dim_{\mathbb{C}} H^q(M, L^k) \le \frac{k^n}{(2\pi)^n}\int_{M(q)} |\det R^L_z| dV_M(z) + o(k^n).$$

Here, the curvature R^L is the Chern curvature of L, which is a global positive real $(1,1)$ form (Locally, $R^L := 2\partial\bar{\partial}\phi$, if we denote the local trivializing section by $s: u\to L$ and $|s(z)|^2_{h^L} = e^{-2\phi(z)}$). Also, $M(q) := \{x\in M : R^L_x$ is non-degenerate and has exactly q eigenvalues$\}$.

Proof. Take the circle bundle $X := \{v\in L^* : |v|^2_{L^*} = 1\}$, which is in fact a CR manifold endowed with a fiber-wise transversal, CR S^1-action. Moreover, by direct computation, locally there is

$$\mathcal{L}_x = \frac{1}{2}R^L_z \text{ for all } x\in X,\ z\in M.$$

With the known fact (for example, see **Theorem 1.4** in Cheng–Hsiao–Tsai [4])

$$\dim_{\mathbb{C}} H^q(M, L^k) = \dim_{\mathbb{C}} H^q_{b,k}(X).$$

Then **Theorem 2.2** gives

$$\begin{aligned}\dim_{\mathbb{C}} H^q(M, L^k) &= \dim_{\mathbb{C}} H^q_{b,k}(X) \\ &\leq \frac{k^n}{2\pi^{n+1}} \int_{X(q)} |\det \mathcal{L}_x| dV_X(x) + o(k^n) \\ &\leq \frac{k^n}{2\pi^{n+1}} \frac{2\pi}{2^n} \int_{M(q)} |\det R_z| dV_M(z) + o(k^n) \\ &= \frac{k^n}{(2\pi)^n} \int_{M(q)} |\det R_z| dV_M(z) + o(k^n).\end{aligned}$$

□

2.3. **The S^1-equivariant Riemann–Roch–Hirzebruch theorem.** We start from some notations and facts about the rigid Hermitian CR geometry.

One one hand, for each $j = 1, \cdots, 2n$, take $\Omega^j_0(X) := \{u \in \bigoplus_{p+q=j} \Omega^{(p,q)}(X) : Tu = 0\}$, and let

$$\Omega^*_0(X) := \bigoplus_{j=0}^{2n} \Omega^j_0(X).$$

Since $dT = Td$, we can again consider the d-subcomplex

$$d : \cdots \to \Omega^j_0(X) \to \Omega^{j+1}_0(X) \to \cdots$$

and the corresponding equivariant cohomology $H^j_{b,0}(X)$.

On the other hand, we say a function u is T-rigid if $Tu = 0$, and a vector bundle F of rank r over X is said to be T-rigid, if X can be covered by some open sets $\{U_j\}_{j=1}^r$ such that the trivializing frames $\{f_{j,k}\}_{k=1}^r$ has T-rigid transition functions. Let $\langle \cdot | \cdot \rangle_F$ be a Hermitian metric on F, then we say it is T-rigid if for every local frame $\{f_j\}_{j=1}^r$, there is $T\langle f_j | f_k \rangle_F$ for all $j, k = 1, \cdots, r$.

Now, fix a T-rigid vector bundle F endowed with a T-rigid fiber metric. It is known that (cf. Cheng–Hsiao–Tsai [4]) there exists a T-rigid connection ∇^F on F such that for any rigid local frame $\{f_j\}_{j=1}^r$ over a open set $D \subset X$, the connection 1-forms $(\theta_{j,k})_{j,k=1}^r$ given by $\nabla^F f_j = f_k \theta_{j,k}$ satisfy $\theta_{j,k=1} \in \Omega^1_0(D)$ for all j, k. Accordingly, we take the T-rigid curvature 2-form

$$\Theta(\nabla^F, F) := d\theta - \theta \wedge \theta$$

and for any real power series $h(z) := \sum_{j=1}^\infty a_j z^j$, $z \in \mathbb{C}$, set

$$H(\Theta(\nabla^F, F)) := \operatorname{Tr}\left(h\left(\frac{i\Theta(\nabla^F, F)}{2\pi} \right) \right) \in \Omega^*_0(X).$$

Then we can check that (cf. Ma–Marinescu [10] and Cheng–Hsiao–Tsai [4])

(1) $H(\Theta(\nabla, F))$ is a closed form.
(2) For two rigid connection ∇ and ∇' on F, then

$$H(\Theta(\nabla, F)) - H(\Theta(\nabla', F)) = dA$$

for some $A \in \Omega^*_0 X$.

Put $h(z) := \log(\frac{z}{1-e^{-z}})$, and take

$$\mathrm{Td}_b(\nabla^F, F) := e^{H(\Theta(\nabla,F))}.$$

Then the tangential Todd class of F is defined by

$$\mathrm{Td}_b(F) := \left[\mathrm{Td}_b(\nabla^F, F)\right] \in H^*_{b,0}(X) := \bigoplus_{j=0}^{2n} H^j_{b,0}(X).$$

Note that $T^{1,0}X$ is a rigid vector bundle over X (cf. Cheng–Hsiao–Tsai [4]).

In Cheng–Hsiao–Tsai [4], they established S^1-equivariant Riemann–Roch–Hirzebruch theorem:

Theorem 2.3. *Assume the same X, then*

$$\sum_{j=1}^{n}(-1)^j \dim_{\mathbb{C}} H^j_{b,m}(X) = \frac{1}{2\pi}\int_X \mathrm{Td}_b(T^{1,0}X) \wedge e^{-\frac{md\omega_0}{2\pi}} \wedge \omega_0.$$

Corollary 2.2 (Riemann-Roch–Hirzebruch theorem)**.** *Let M be a compact Hermitian manifold with $\dim_{\mathbb{C}} M = n$, (L, h^L) be a holomorphic line bundle over M with a smooth hermitian metric. Then for all $k \in \mathbb{N}$,*

$$\sum_{j=1}^{n}(-1)^j \dim_{\mathbb{C}} H^j(M, L^k) = \int_M \mathrm{Td}(T^{1,0}M) \wedge ch(L^k).$$

Proof. As before, take $X := \{v \in L^* : |v|^2_{L^*} = 1\}$, and then

$$\begin{aligned}\sum_{j=1}^{n}(-1)^j \dim_{\mathbb{C}} H^j(M, L^k) &= \sum_{j=1}^{n}(-1)^j \dim_{\mathbb{C}} H^j_{b,k}(X)\\ &= \frac{1}{2\pi}\int_X \mathrm{Td}_b(T^{1,0}X) \wedge e^{-\frac{kd\omega_0}{2\pi}} \wedge \omega_0\\ &= \int_M \mathrm{Td}(T^{1,0}M) \wedge \mathrm{ch}(L^k).\end{aligned}$$

□

3. The proof of the S^1-equivariant weak CR Morse inequality

In this section, we give a complete survey on Hsiao–Li [9]. Some idea and result involved here will be in the used later section.

3.1. **The BRT trivialization.** Let X be a compact connected CR manifold endowed with a CR, transversal S^1-action, then as in Baouendi–Rothschild–Trèves [1], after applying Newlander-Nirenberg's theorem by the integrability assumption and the CR, transversal conditions for the S^1-action, we can see that locally X is a Heisenberg group. We summarize this fact along with some related results in Hsiao–Li [9] as follows:

Theorem 3.1. *Assume the same X, then*

(1) *For all $x_0 \in X$, there exists $\epsilon, \delta > 0$, canonical coordinate patch near x_0*

$$D := \{(z, \theta) : |z| < \epsilon, |\theta| < \delta\}$$

and a local coordinate

$$(x_1, x_2, \cdots, x_{2n-1}, x_{2n}, x_{2n+1}) = (z_1, \cdots, z_n, \theta),$$

where $z_j := x_{2j-1} + ix_{2j}$, $\theta := x_{2n+1}$, *such that*

(a) *the fundamental vector field induced by* S^1*-action is in the form*

$$T = \frac{\partial}{\partial\theta}.$$

(b) *We can find a* $\phi(z) \in \mathscr{C}^\infty(D, \mathbb{R})$ *such that*

$$\left\{ Z_j := \frac{\partial}{\partial z_j} + i\frac{\partial\phi(z)}{\partial z_j}\frac{\partial}{\partial\theta} \right\}_{j=1}^n$$

forms a basis of $T_x^{1,0}X$ *for all* $x \in D$.

(c) *Follow the notations above, we can take the pair* (z, θ, ϕ) *such that*

$$\Big(z(x_0), \theta(x_0) \Big) = (0, 0)$$

and

$$\phi(z) = \sum_{j=1}^n \lambda_j |z_j|^2 + O(|z|^3) \text{ for all } (z, \theta) \in D,$$

where $\{\lambda_j\}_{j=1}^n$ *are eigenvalues of the Levi form of* X *at* x_0 *with respect to the chosen* T*-rigid metric.*

We call such pair (z, θ, ϕ) *is trivial at* x_0.

(2) *Let* $x_0 \in X_{\text{reg}}$, *then there exists an* $\epsilon_0 > 0$ *and a pair* (z, θ, ϕ) *in*

$$D := \{(z, \theta) : |z| < \epsilon_0, |\theta| < \pi\}$$

trivial at x_0.

(3) *Let* $x_0 \in X_k$, $k > 1$, *then for all* $\epsilon > 0$, *there exists an* $\epsilon_0 > 0$ *and a pair* (z, θ, ϕ) *in*

$$D_\epsilon := \left\{ (z, \theta) : |z| < \epsilon_0, |\theta| < \frac{\pi}{k} - \epsilon \right\}$$

trivial at x_0.

Corollary 3.1. *Let* T *be the fundamental vector filed induced by the* S^1*-action, then* $T\bar{\partial}_b = \bar{\partial}_b T$

Proof. Locally, in the coordinate patch, for $u \in \Omega^{(0,q)}(X)$,

$$Tu = \frac{\partial u}{\partial\theta}$$

and

$$\bar{\partial}_b u = \sum_{\substack{|J|=q \\ 1\leq j\leq n}}{}' \left(\frac{\partial u_J}{\partial \bar{z}_j} + i\frac{\partial\phi(z)}{\partial\bar{z}_j}\frac{\partial u_J}{\partial\theta} \right) d\bar{z}_j \wedge d\bar{z}^J.$$

Since the term concerning ϕ is independent of θ, it's clear that T commutes with $\bar{\partial}_b$. $\square$

To give the Fourier decomposition of the smooth $(0, q)$ form via $-iT$, we need some knowledge about the rigid geometry: A vector field V on D is said to be T-rigid if

$$(de^{i\theta})_x V(x) = V(e^{i\theta} \circ x).$$

In fact, there exists a so called T-rigid metric $\langle\cdot|\cdot\rangle$ on $\mathbb{C}TX$ satisfying

(1) For T-rigid vector field V, W on D,

$$\langle V(x)|W(x)\rangle = \left\langle (de^{i\theta})_x V(x)\middle|(de^{i\theta})_x W(x)\right\rangle = \left\langle V(e^{i\theta}\circ x)\middle|W(e^{i\theta}\circ x)\right\rangle.$$

(2) $T^{1,0}X \perp T^{0,1}X$ and $T \perp (T^{1,0}X \oplus T^{0,1}X)$.

(3) If V, W are real vector filed, then $\langle V|W\rangle \in \mathbb{R}$.

For such T-rigid metric, and any $x_0 \in X$, take the canonical coordinate patch

$$D := \tilde{D} \times (-\delta, \delta)$$

near x_0 such that the pair (z, θ, ϕ) trivial at x_0. We identify the patch as an open subset of $\mathbb{C}^n \times \mathbb{R}$, then there exists an orthonormal frame $\{e_j\}_{j=1}^n$ of $T^{*0,1}D$ with respect to the fixed T-rigid Hermitian metric so that

(1) $e_j(x) = e_j(z) = d\bar{z}_j + O(|z|)$.

(2) The volume form of X respect to the fixed T-rigid Hermitian metric on $\mathbb{C}TX$ is of the form

$$dV_X(x) = \lambda(z)dv(z)d\theta,$$

where $\lambda(z) \in \mathscr{C}^\infty(\tilde{D}, \mathbb{R})$ is independent of θ and $dv(z) = 2^n dx_1 \cdots dx_{2n}$.

With this facts, we can now present the S^1-Fourier decomposition.

Proposition 3.1. *We can decompose the space orthogonally:*

(1) $\Omega^{(0,q)}(X) = \oplus_{m\in\mathbb{Z}}\Omega_m^{(0,q)}X$.

(2) $L^2_{(0,q)}(X) = \oplus_{m\in\mathbb{Z}}L^2_{(0,q),m}(X)$.

Proof. (1) For $u \in \Omega^{(0,q)}(X)$, $\theta \in (-\pi, \pi)$, we have $\int_{-\pi}^{\pi} u(e^{i\theta}\circ x)e^{-im\theta}d\theta = O(\frac{1}{m^2})$ by applying integration by parts twice. So as in the case of S^1 Fourier series,

$$u(x) = \sum_{m\in\mathbb{Z}} \frac{1}{2\pi}\int_{-\pi}^{\pi} u(e^{i\theta}\circ x)e^{-im\theta}d\theta \text{ in } C^\infty \text{ topology.}$$

On one hand, let

$$(Q_m^{(q)}u)(x) := \frac{1}{2\pi}\int_{-\pi}^{\pi} u(e^{i\theta}\circ x)e^{-im\theta}d\theta,$$

then there is

$$(Q_m^{(q)}u)(e^{i\phi}\circ x) = e^{im\phi}(Q_m^{(q)}u)(x).$$

for all $\phi \in [0, 2\pi)$, i.e. $Q_m^{(q)}u \in \Omega_m^{(0,q)}(X)$.

On the other hand, for $u \in \Omega$ and $v \in \Omega_n^{(0,q)}X$, where $m \neq n$. Then because $(\cdot|\cdot)$ is T-rigid, locally there is

$$\begin{aligned}(-iTu|v) &= \int_D \left(-i\frac{\partial}{\partial\theta}u\right)\bar{v}2^n\lambda(x_1,\cdots,x_{2n})dx_1\cdots dx_{2n}d\theta \\ &= \int_D u\left(-i\frac{\partial}{\partial\theta}\bar{v}2^n\lambda(x_1,\cdots,x_{2n})\right)dx_1\cdots dx_{2n}d\theta \\ &= \int_D u\overline{-i\frac{\partial}{\partial\theta}v}2^n\lambda(x_1,\cdots,x_{2n})dx_1\cdots dx_{2n}d\theta \\ &= (u|-iTv).\end{aligned}$$

In other words, we have

$$m(u|v) = (-iTu|v) = (u|-iTv) = n(u|v),$$

i.e. $(u|v) = 0$. In conclusion, $Q_m^{(q)} : u \in \Omega^{(0,q)}(X) \mapsto \frac{1}{2\pi}\int_{-\pi}^{\pi} u(e^{i\theta} \circ x)e^{-im\theta}d\theta \in \Omega_m^{(0,q)}(X)$ is an orthogonal projection, and the Fourier decomposition is orthogonal with respect to T-rigid $(\cdot|\cdot)$.

(2) Just take the completion of $\Omega_m^{(0,q)}X$ with respect to T- rigid $(\cdot|\cdot)$.

$\square$

3.2. **The Hodge theorem for Kohn–Rossi Laplacian.** In this section, we follow the argument appeared in Cheng–Hsiao–Tsai [4] to gain the Hodge theorem for Kohn–Rossi Laplacian. It's well known that $\square_b^{(q)} := \bar\partial_b\bar\partial_b^* + \bar\partial_b^*\bar\partial_b$, where $\bar\partial_b^*$ is the formal adjoint with respect to the T-rigid inner product $(\cdot|\cdot)$, may not be elliptic, $\square_{b,m}^{(q)}$ neither. One classical method to establish the corresponding Hodge theorem is to use Kohn's subelliptic estimate on $Y(q)$ condition. (cf. Chen–Shaw [3], Theorem 8.4.2). However, under the assumption that X is a compact CR manifold admitting a transversal CR S^1-action, we can consider the auxiliary differential operator $\Delta_{b,m}^{(q)} := \square_{b,m}^{(q)} - T^2$, which is elliptic, because locally after choosing an orthonormal basis $\{L_j\}_{j=1}^n$ of $T_x^{(0,1)}X$, we have the expression

$$\sigma_{\Delta_{b,m}^{(q)}}(x,\xi) = \sum_{j=1}^n |\sigma_{L_j}(x,\xi)|^2 - \sigma_T(x,\xi)^2 > 0.$$

(Recall that the principal symbol is coordinate invariant, so we can apply the transversal property for the circle action, which implies T is non-vanishing, to make $T = \frac{\partial}{\partial\theta}$, then $\sigma_T = i\xi$, i.e. $\sigma_T^2 = -\xi^2 < 0$ for any nonzero ξ).

Accordingly, we show how to deduce the Hodge theorem of $\square_{b,m}^{(q)}$ via $\Delta_{b,m}^{(q)}$. (The argument is quite standard, and it works for general elliptic formally self adjoint operator).

Theorem 3.2.

(1) *Consider the extension*

$$\square_{b,m}^{(q)} : \mathrm{Dom}\ \square_{b,m}^{(q)} \subset L^2_{(0,q),m}(X) \to L^2_{(0,q),m}(X)$$

by

$$\mathrm{Dom}\ \square_{b,m}^{(q)} := \{u \in L^2_{(0,q),m}(X) : \square_{b,m}^{(q)}u \in L^2_{(0,q),m}(X)\}.$$

Then such extension is self adjoint, which means:

(3.1) $$\left(\square_{b,m}^{(q)}u\middle|v\right) = \left(u\middle|\square_{b,m}^{(q)}v\right) \text{ for all } u,v \in \mathrm{Dom}\ \square_{b,m}^{(q)} \cap \mathrm{Dom}\ \square_{b,m}^{(q),*}.$$

(3.2) $$\mathrm{Dom}\square_{b,m}^{(q)} = \mathrm{Dom}\ \square_{b,m}^{(q),*}$$

where

$$\mathrm{Dom}\ \square_{b,m}^{(q),*} := \{v \in L^2_{(0,q),m}(X) : \forall u \in \mathrm{Dom}\ \square_{b,m}^{(q)},\ \exists\, c > 0 \text{ such that } \left|\left(\square_{b,m}^{(q)}u|v\right)\right| < c\|u\|_{L^2}\}.$$

(2) Spec $\Box_{b,m}^{(q)}$ *consists only of eigenvalues.*

Here, for $\lambda \in \mathrm{Spec}\,\Box_{b,m}^{(q)}$, *we mean the map*

$$\lambda - \Box_{b,m}^{(q)} : \mathrm{Dom}\,\Box_{b,m}^{(q)} \subset L^2_{(0,q),m}(X) \to L^2_{(0,q),m}(X)$$

is either

(a) *not injective,*

(b) *or injective but not surjective,*

(c) *or bijective but the inverse map is unbounded.*

(3) Spec $\Box_{b,m}^{(q)}$ *is a discrete subspace of* $[0,\infty)$, *and for* $\lambda \in \mathrm{Spec}\,\Box_{b,m}^{(q)}$, *the eigenspace*

$$\mathcal{H}_{b,m,\lambda}^{(q)}(X) := \{u \in \mathrm{Dom}\,\Box_{b,m}^{(q)} : \Box_{b,m}^{(q)} u = \lambda u\}.$$

is a finite dimensional subspace of $\Omega_m^{(0,q)}(X)$. *Moreover, the harmonic form*

$$\mathcal{H}_{b,m}^{(q)}(X) := \mathcal{H}_{b,m,0}^{(q)}(X)$$

is isomorphic to the m-th Kohn-Rossi cohomology, i.e.

$$\mathcal{H}_{b,m}^{q}(X) \cong H_{b,m}^{q}(X). \tag{3.3}$$

Proof.

(1) First of all, we use the basic elliptic regularity of $\Delta_{b,m}$ to claim:

$$\mathrm{Dom}\,\Box_{b,m}^{(q)} = H^2_{(0,q),m}(X). \tag{3.4}$$

For the side $\mathrm{Dom}\,\Box_{b,m}^{(q)} \supset H^2_{(0,q),m}(X)$, it is clear by the continuity of the second order differential operator $\Box_{b,m}^{(q)} : H^2_{(0,q),m}(X) \to L^2_{(0,q),m}(X)$. Conversely, consider the auxiliary elliptic differential operator

$$\Delta_{b,m}^{(q)} := \Box_{b,m}^{(q)} - T^2,$$

then by ellipticity we can construct the parametrix Q as in Grigis–Sjöstrand [7, Theorem 4.1], such that

$$Q\Delta_{b,m}^{(q)} = Id - S$$

where Q is a pseudodifferential operator of order -2, and S is a smoothing operator. Then for $u \in \mathrm{Dom}\,\Box_{b,m}^{(q)}$, clearly

$$\begin{aligned}\|u\|_{H^2} &\le \|Q(\Delta_{b,m}^{(q)} u)\|_{H^2} + \|Su\|_{H^2} \\ &\le \|Q(\Box_{b,m}^{(q)} u)\|_{H^2} + m^2\|Qu\|_{H^2} + \|Su\|_{H^2}.\end{aligned}$$

Since the pseudodifferential operators Q and S act continuously on the Sobolev space, i.e.

$$Q : H^s_{(0,q),m}(X) \to H^{s+2}_{(0,q),m}(X),\ S : H^s_{(0,q),m}(X) \to H^t_{(0,q),m}(X) \text{ for all } s,t \in \mathbb{R}$$

are continuous, we can find that $u \in H^2_{(0,q),m}(X)$. So (3.4) holds.

By the claim (3.4), the symmetry condition (3.1) holds, because we can accordingly pick an approximation

$$v_j \in \Omega_m^{(0,q)}(X) \to v \in \mathrm{Dom}\,\Box_{b,m}^{(q)} \cap \mathrm{Dom}\,\Box_{b,m}^{(q),*} \text{ in } H^2_{(0,q),m}(X)$$

such that

$$\Box^{(q)}_{b,m} v_j \in \Omega^{(0,q)}_m(X) \to \Box^{(q)}_{b,m} v \text{ in } L^2_{(0,q),m}(X).$$

This implies

$$\left(\Box^{(q)}_{b,m} u \middle| v\right) = \lim_{j\to\infty}\left(\Box^{(q)}_{b,m} u \middle| v_j\right),\ \left(u \middle| \Box^{(q)}_{b,m} v\right) = \lim_{j\to\infty}\left(u \middle| \Box^{(q)}_{b,m} v_j\right),$$

and by the definition for the derivative of distribution, $\left(\Box^{(q)}_{b,m} u \middle| v_j\right) = \left(u \middle| \Box^{(q)}_{b,m} v_j\right)$. So (3.1) follows.

As for the (3.2), observe that Dom $\Box^{(q),*}_{b,m}$ collects all $v \in L^2_{(0,q),m}(X)$ such that the map

$$u \in \text{Dom } \Box^{(q)}_{b,m} \mapsto \left(\Box^{(q)}_{b,m} u \middle| v\right)$$

is bounded linear. By Riesz's lemma, there exists $w \in L^2_{(0,q),m}(X)$ s.t.

$$\left(\Box^{(q)}_{b,m} u | v\right) = (u|w)$$

and this w is denoted by $\Box^{(q),*}_{b,m} v$. Hence, by the symmetric conditions (3.1) and Cauchy–Schwarz inequality, for $v \in \text{Dom } \Box^{(q)}_{b,m}$ and all $u \in \text{Dom } \Box^{(q)}_{b,m}$,

$$\left|\left(\Box^{(q)}_{b,m} u \middle| v\right)\right| = \left|\left(u \middle| \Box^{(q)}_{b,m} v\right)\right| \leq \|u\|_{L^2} \cdot \|\Box^{(q)}_{b,m} v\|_{L^2} < c\|u\|_{L^2},$$

for some $0 < c < \infty$. So Dom $\Box^{(q)}_{b,m} \subset$ Dom $\Box^{(q),*}_{b,m}$. For another side of inclusion, we verify Dom $\Box^{(q),*}_{b,m} \subset H^2_{(0,q),m}(X)$, then we can find that $\Box^{(q)}_{b,m}$ is self-adjoint by (3.4). Recall that by Riesz's lemma, $v \in$ Dom $\Box^{(q),*}_{b,m}$ if and only if there is $w \in L^2_{(0,q),m}(X)$ such that for all $u \in$ Dom $\Box^{(q)}_{b,m}$

$$\left(\Box^{(q)}_{b,m} u \middle| v\right) = (u|w).$$

Since $\Omega^{(0,q)}_m(X) \subset$ Dom $\Box^{(q)}_{b,m}$, this implies that

$$\Box^{(q)}_{b,m} v = w \in L^2_{(0,q),m}(X) \text{ where we view } v \text{ as a distribution.}$$

So by the same a priori estimate for the elliptic operator $\Delta^{(q)}_{b,m}$ used earlier, we get

$$v \in H^2_{(0,q),m}(X)$$

and the claim follows.

(2) First, we argue Spec $\Box^{(q)}_{b,m}$ consists only of eigenvalues: suppose $\lambda \in$ Spec $\Box^{(q)}_{b,m}$, $\lambda - \Box^{(q)}_{b,m}$ is injective, we claim that $\lambda - \Box^{(q)}_{b,m}$ is also surjective and has a bounded inverse, contradicting the definition of spectrum. In fact, if $\lambda - \Box^{(q)}_{b,m}$ is injective, we can observe that:

(3.5) There exists $C > 0$ such that $\left\|(\lambda - \Box^{(q)}_{b,m})u\right\|_{L^2} > C\|u\|_{L^2}$ for all $u \in$ Dom $\Box^{(q)}_{b,m}$.

$$\overline{\text{Rang } (\lambda - \Box^{(q)}_{b,m})} = \text{Rang } (\lambda - \Box^{(q)}_{b,m}). \tag{3.6}$$

$$\text{Rang } (\lambda - \Box^{(q)}_{b,m})^{\perp} = \ker(\lambda - \Box^{(q),*}_{b,m}). \tag{3.7}$$

Here, Rang $(\lambda-\Box^{(q)}_{b,m}) := \left\{(\lambda-\Box^{(q)}_{b,m})u : u\in \text{Dom}\,\Box^{(q)}_{b,m}\right\}$. The contradiction follows by combining the standard Hilbert space theory, (3.6), (3.7) and $\Box^{(q)}_{b,m}=\Box^{(q),*}_{b,m}$ to get

$$\begin{aligned}L^2_{(0,q),m}(X) &= \overline{\text{Rang}\,(\lambda-\Box^{(q)}_{b,m})}\oplus \text{Rang}\,(\lambda-\Box^{(q)}_{b,m})^{\perp}\\ &= \text{Rang}\,(\lambda-\Box^{(q)}_{b,m}).\end{aligned}$$

So $(\lambda-\Box^{(q)}_{b,m})^{-1}$ exists. However, (3.5) implies the inverse of the resolvent is bounded.

Now, we go back to prove these three observations. For (3.5), suppose it is not true, i.e. for all $j\in\mathbb{N}$, then we can find $u_j\in\text{Dom}\,\Box^{(q)}_{b,m}$, $\|u_j\|_{L^2}=1$ such that

$$\left\|(\lambda-\Box^{(q)}_{b,m})u_j\right\|_{L^2}<\frac{1}{j}\|u_j\|_{L^2}=\frac{1}{j}.$$

Then, again construct the parametrix of the elliptic operator

$$\lambda-\Delta^{(q)}_{b,m}:=\lambda-\Box^{(q)}_{b,m}+T^2$$

(it's elliptic because $\sigma_{\lambda-\Delta^{(q)}_{b,m}}(x,\xi)=-\sigma_{\Delta^{(q)}_{b,m}}(x,\xi)\neq 0$), then there is a $C'>0$ such that

$$\begin{aligned}\|u_j\|_{H^2} &< C'\left(\left\|(\lambda-\Delta^{(q)}_{b,m})u_j\right\|_{L^2}+\|u_j\|_{L^2}\right)\\ &\le C'\left(\left\|(\lambda-\Box^{(q)}_{b,m})u_j\right\|_{L^2}+m^2\|u_j\|_{L^2}+\|u_j\|_{L^2}\right)\\ &\le C'' : \text{a positive constant.}\end{aligned}$$

By Rellich's lemma, the inclusion map

$$H^2_{(0,q),m}(X)\hookrightarrow L^2_{(0,q),m}(X)$$

is compact. So there is a subsequence $\{u_{j_k}\}$ of $\{u_j\}$ such that

$$u_{j_k}\to u\in L^2_{(0,q),m}(X)$$

with $\|u\|_{L^2}=1$.

However, for all $v\in\Omega^{(0,q)}_m(X)$,

$$\begin{aligned}\left|\left((\lambda-\Box^{(q)}_{b,m})u_j\,\middle|\,v\right)\right| &= \left|\left(u_j\,\middle|\,(\lambda-\Box^{(q)}_{b,m}v)\right)\right|\\ &\le \|u_j\|_{L^2}\cdot\left\|(\lambda-\Box^{(q)}_{b,m})v\right\|_{L^2}\\ &\le \frac{C'''}{j}\text{ for some } C'''>0\\ &\to 0 \text{ for derivative in the distribution sense.}\end{aligned}$$

But we also have

$$\left((\lambda-\Box^{(q)}_{b,m})u\,\middle|\,v\right)=\left(u\,\middle|\,(\lambda-\Box^{(q)}_{b,m}v)\right)=\lim_{j\to\infty}\left(u_j\,\middle|\,(\lambda-\Box^{(q)}_{b,m}v)\right)$$

by the self-adjointness of $\Box^{(q)}_{b,m}$ and the completeness of the space of distributions. In conclusion, $u\in\ker(\lambda-\Box^{(q)}_{b,m})$ contradicting the assumption that $\lambda-\Box^{(q)}_{b,m}$ is injective, so the estimate (3.5) holds.

For (3.6), given $v_j \in \text{Rang } \Box^{(q)}_{b,m}$, $v_j \to v \in L^2_{(0,q),m}(X)$, we have to show

$$v \in \text{Rang } (\lambda - \Box^{(q)}_{b,m}).$$

Rewrite $v_j = (\lambda - \Box^{(q)}_{b,m})u_j$ for $u_j \in \text{Dom } \Box^{(q)}_{b,m}$, then by the estimate (3.5)

$$\left\|(\lambda - \Box^{(q)}_{b,m})u\right\|_{L^2} > C\|u\|_{L^2} \text{ for all } u \in \text{Dom } \Box^{(q)}_{b,m},$$

we know $\{u_j\}_{j=1}^{\infty}$ form a Cauchy sequence in $L^2_{(0,q),m}(X)$.

So, $u_j \to u \in L^2_{(0,q),m}(X)$ and $(\lambda - \Box^{(q)}_{b,m})u = v$, i.e. $v \in \text{Rang } (\lambda - \Box^{(q)}_{b,m})$. Hence, the closed range property (3.6) is also true.

Finally, for (3.7), we check this by the following two claims:

$$\ker(\lambda - \Box^{(q),*}_{b,m}) \text{ is a finite dimensional subspace of } \Omega^{(0,q)}_m(X). \tag{3.8}$$

$$\text{Rang } (\lambda - \Box^{(q)}_{b,m}) = \ker(\lambda - \Box^{(q),*}_{b,m})^{\perp}. \tag{3.9}$$

If (3.8), (3.9) are true, then by the basic linear algebra, we get (3.7) by

$$\text{Rang } (\lambda - \Box^{(q)}_{b,m})^{\perp} = \left(\ker(\lambda - \Box^{(q),*}_{b,m})^{\perp}\right)^{\perp} = \ker(\lambda - \Box^{(q),*}_{b,m}).$$

For (3.8), on one hand, by the regularity of $\Delta^{(q)}_{b,m}$ and $\Box^{(q)}_{b,m} = \Box^{(q),*}_{b,m}$, we have

$$\ker(\lambda - \Box^{(q),*}_{b,m}) = \ker(\lambda - \Box^{(q)}_{b,m}) \subset \Omega^{(0,q)}_m(X).$$

On the other hand, suppose $\ker(\lambda - \Box^{(q)}_{b,m})$ is infinite dimensional, take an orthonormal basis $\{u_j\}_{j=1}^{\infty}$ for $\ker(\lambda - \Box^{(q)}_{b,m})$, then the a priori estimate of $\Delta^{(q)}_{b,m}$ suggests that there are C, $C' > 0$ such that

$$\|u_j\|_{H^2} \leq C\left(\lambda\|u_j\|_{L^2} + \left\|\Box^{(q)}_{b,m}u_j\right\|_{L^2} + m^2\|u_j\|_{L^2}\right) \leq C'$$

Therefore, Rellich's lemma gives a subsequence $\{u_{j_k}\}_{k=1}^{\infty}$ satisfying

$$u_{j_k} \to u \in L^2_{(0,q),m}(X).$$

But $\|u_j - u_k\|^2_{L^2} = 2$, so the L^2 norm convergence is impossible.

As for (3.9), first note that the side

$$\text{Rang } (\lambda - \Box^{(q)}_{b,m}) \subset \ker(\lambda - \Box^{(q),*}_{b,m})^{\perp}$$

is clear.

Suppose

$$\text{Rang } (\lambda - \Box^{(q)}_{b,m}) \subsetneq \ker(\lambda - \Box^{(q),*}_{b,m})^{\perp},$$

then we can find a $y_0 \in L^2_{(0,q),m}(X)$ such that

$$y_0 \in \ker(\lambda - \Box^{(q),*}_{b,m})^{\perp},\ y_0 \notin \text{Rang } (\lambda - \Box^{(q)}_{b,m}) = \overline{\text{Rang } (\lambda - \Box^{(q)}_{b,m})}.$$

For all $y \in \text{Rang } (\lambda - \Box^{(q)}_{b,m})$, consider a linear subspace W of $L^2_{(0,q),m}(X)$, which is generated by y_0 and $\text{Rang } (\lambda - \Box^{(q)}_{b,m})$, and take a linear map

$$f(y + ty_0) := t \in \mathbb{C} \text{ on } W.$$

Since $y_0 \notin \overline{\text{Rang }(\lambda - \Box^{(q)}_{b,m})}$, there is $\delta > 0$ such that $\|z - y_0\|_{L^2} > \delta$ for all $z \in$ Rang $(\lambda - \Box^{(q)}_{b,m})$. We can hence find that f is bounded linear, for its operator norm

$$||f||_{W^*} := \inf\{c \geq 0 : \|f(y + ty_0)\|_{L^2} \leq c|t|\} \leq \frac{|t|}{\|y + ty_0\|_{L^2}} = \frac{1}{\| - \frac{y}{t} - y_0\|_{L^2}} < \frac{1}{\delta}.$$

Apply the Hahn-Banach theorem, we have an extension

$$\tilde{f} : L^2_{(0,q),m}(X) \to \mathbb{C}$$

with

$$\tilde{f}(y_0) = 1 \text{ by taking } y = 0,\ t = 1$$

and

$$tildef(y) = 0 \text{ for all } y \in \text{Rang } (\lambda - \Box^{(q)}_{b,m}) \text{ by taking } t = 0.$$

By Riesz's lemma, there exists $\tilde{y}_0 \in L^2_{(0,q),m}(X)$ such that

$$f(y) = (y|\tilde{y}_0).$$

In other words, there are

$$(y_0|\tilde{y}_0) = 1$$

and

$$(y|\tilde{y}_0) = 0 \text{ for all } y \in \text{Rang } (\lambda - \Box^{(q)}_{b,m}).$$

However, by $\Box^{(q)}_{b,m} = \Box^{(q),*}_{b,m}$, the second equation means that

$$\forall u \in \text{Dom } (\lambda - \Box^{(q)}_{b,m}),\ \left((\lambda - \Box^{(q)}_{b,m})u\middle|\tilde{y}_0\right) = \left(u\middle|(\lambda - \Box^{(q)}_{b,m})\tilde{y}_0\right) = 0.$$

In particular, this holds for all $u \in \Omega^{(0,q)}_m(X)$, so $\tilde{y}_0 \in \ker(\lambda - \Box^{(q)}_{b,m}) = \ker(\lambda - \Box^{(q),*}_{b,m})$, and hence

$$(y_0|\tilde{y}_0) = 0.$$

This leads to a contradiction.

(3) Since Spec $\Box^{(q)}_{b,m}$ consists only by eigenvalues, and in fact $\Box^{(q)}_{b,m}$ is also a positive operator, so

$$\lambda\|u\|^2_{L^2} = \left(\Box^{(q)}_{b,m}u\middle|u\right) = (\|\bar{\partial}_b u\|^2_{L^2} + \|\bar{\partial}^*_b u\|^2_{L^2}) \geq 0,$$

i.e. Spec $\Box^{(q)}_{b,m} \subset [0, \infty)$. As for the discreteness of Spec $\Box^{(q)}_{b,m}$ and that $\mathcal{H}^{(q)}_{b,m,\lambda}(X)$ is finite dimensional, they follow from the same argument of illustrating $\ker(\lambda - \Box^{(q)}_{b,m})$ is finite dimensional. Also, $\mathcal{H}^{(q)}_{b,m,\lambda}(X) \subset \Omega^{(0,q)}_m X$ comes form the same regularity trick of $\Delta^{(q)}_m$ earlier.

It remains to prove (3.3). Before starting, we introduce the idea of partial inverse (or the so called Green's operator) of $\Box^{(q)}_{b,m}$. In fact, we can find such operator

$$N^{(q)}_m : L^2_{(0,q),m}(X) \to \text{Dom } \Box^{(q)}_{b,m}$$

satisfying

$$N^{(q)}_m \Box^{(q)}_{b,m} = Id - \Pi^{(q)}_m \text{ on Dom}\Box^{(q)}_{b,m}$$

and

$$\Box^{(q)}_{b,m} N^{(q)}_m = Id - \Pi^{(q)}_m \text{ on } L^2_{(0,q),m}X,$$

where

$$\Pi_m^{(q)} : \mathrm{Dom}\,\Box_{b,m}^{(q)} \to \ker \Box_{b,m}^{(q)} \text{ is the orthogonal projection.}$$

The existence of such $N_m^{(q)}$ is as follows: consider the bijective map

$$\Box_{b,m}^{(q)} : \mathrm{Dom}\,\Box_{b,m}^{(q)} \cap \left(\ker \Box_{b,m}^{(q)}\right)^{\perp} \to \mathrm{Rang}\,\Box_{b,m}^{(q)}.$$

Then by the bounded inverse theorem, we have a bounded linear map

$$\tilde{N}_m^{(q)} : \mathrm{Rang}\,\Box_{b,m}^{(q)} \to \mathrm{Dom}\,\Box_{b,m}^{(q)} \cap \left(\ker \Box_{b,m}^{(q)}\right)^{\perp}.$$

Consider the extension

$$N_m^{(q)} := \begin{cases} \tilde{N}_m^{(q)} : \text{ on } \mathrm{Rang}\,\Box_{b,m}^{(q)} \\ 0 : \text{ on } (\mathrm{Rang}\,\Box_{b,m}^{(q)})^{\perp} = \ker \Box_{b,m}^{(q)} \end{cases}.$$

In this way, it's clear that the operator is what we want.

Now, we are ready to prove the final part of the theorem, i.e.

$$\mathcal{H}_{b,m}^{q}(X) = H_{b,m}^{q}(X) := \frac{\ker \bar{\partial}_{b,m}^{(q)}}{\mathrm{Im}\bar{\partial}_{b,m}^{(q-1)}}$$

where

$$\ker \bar{\partial}_{b,m}^{(q)} := \{u \in \Omega_m^{(0,q)}(X) : \bar{\partial}_{b,m}^{(q)} u = 0\}$$

and

$$\mathrm{Im}\bar{\partial}_{b,m}^{(q-1)} := \{\bar{\partial}^{(q-1)} u \in \Omega_m^{(0,q)}(X) : u \in \Omega_m^{(0,q-1)}(X)\}.$$

Consider the map

$$\tau : \ker \bar{\partial}_{b,m}^{(q)} \to \ker \Box_{b,m}^{(q)}$$

by $\tau(u) := \Pi_m^{(q)} u$. Then it suffices to show $\ker \tau = \mathrm{Im}\bar{\partial}_{b,m}^{(q-1)}$.

Note that $\mathrm{Im}\bar{\partial}_{b,m}^{(q-1)} \subset \ker \tau$ is clear by $\ker \Box_{b,m}^{(q)} \subset \ker \bar{\partial}_{b,m}^{(q)}$. For another inclusion $\ker \tau \subset \mathrm{Im}\bar{\partial}_{b,m}^{(q-1)}$, we need the idea of partial inverse introduced earlier. If $u \in \ker \bar{\partial}_{b,m}^{(q)}$ and $\Pi_m^{(q)} u = 0$, then

$$\begin{aligned} u &= u - \Pi_m^{(q)} u \\ &= \Box_{b,m}^{(q)} N_m^{(q)} u \\ &= \bar{\partial}_{b,m}^{(q),*} \bar{\partial}_{b,m}^{(q)} + \bar{\partial}_{b,m}^{(q)} \bar{\partial}_{b,m}^{(q),*}) N_m^{(q)} u \\ &= \bar{\partial}_{b,m}^{(q)} (\bar{\partial}_{b,m}^{(q),*} N_m^{(q)} u) \end{aligned}$$

The last equality holds because

$$
\begin{aligned}
(\bar\partial^{(q),*}_{b,m}\bar\partial^{(q)}_{b,m}N^{(q)}_m|\bar\partial^{(q),*}_{b,m}\bar\partial^{(q)}_{b,m}N^{(q)}_m u) &= (\bar\partial^{(q),*}_{b,m}\bar\partial^{(q)}_{b,m}\bar\partial^{(q),*}_{b,m}\bar\partial^{(q)}_{b,m}N^{(q)}_m u|N^{(q)}_m u)\\
&= (\bar\partial^{(q),*}_{b,m}\bar\partial^{(q)}_{b,m}\Box^{(q)}_{b,m}N^{(q)}_m u|N^{(q)}_m u)\\
&\text{(note that } \bar\partial^{(q)}_{b,m}\bar\partial^{(q)}_{b,m}=0)\\
&= (\bar\partial^{(q),*}_{b,m}\bar\partial^{(q)}_{b,m}(Id-\Pi^{(q)}_m)u|N^{(q)}_m u)\\
&= (\bar\partial^{(q),*}_{b,m}\bar\partial^{(q)}_{b,m}u|N^{(q)}_m u)\\
&= 0.
\end{aligned}
$$

Finally, we demonstrate indeed $u \in \mathrm{Im}\bar\partial^{(q-1)}_{b,m}$.

Observe that if $N^{(q)}_m u := v$ for $u \in \ker \bar\partial^{(q)}_{b,m} \subset \Omega^{(0,q)}_m(X)$, then

$$\Box^{(q)}_{b,m} v = u - \Pi^{(q)}_m u = u \in L^2_{(0,q),m}(X)$$

So the elliptic regularity of $\Delta^{(q)}_{b,m}$ gives $v \in \Omega^{(0,q)}_m(X)$, and $\bar\partial^{(q),*}_{b,m}N^{(q)}_m u = \bar\partial^{(q),*}_{b,m} v \in \Omega^{(0,q)}_m(X)$.

□

3.3. **A Scaling Technique.** In this part, we introduce the idea of semi-classical approximation of the scaled Laplcian by the flat Laplcian on $\mathbb{C}^n$. First of all, we fix a $x \in X$, and take the canonical patch

$$D = \tilde D \times (-\delta,\delta) := \{(z,\theta) : |z| < \epsilon, |\theta| < \delta\}$$

as in **Theorem 3.1**, and we identify $\tilde D$ as an open set in $\mathbb{C}^n$. Take the weighted L^2 inner product on $\Omega^{(0,q)}_0 \tilde D$ by

$$(f|g)_{2m\phi} := \int_{\tilde D} \langle f|g\rangle e^{-2m\phi(z)}\lambda(z)dv(z)$$

where $\lambda(z)$ is the real-valued smooth function in **Theorem 3.1**.

Now, assume m is large enough such that $\tilde D_{\log m}$ is bounded open in $\tilde D$, and take the scaled map

$$F_m : z \in \tilde D_{\log m} \mapsto \frac{z}{\sqrt m} \in \tilde D$$

where $\tilde D_{\log m} := \{z \in \tilde D : |z_j| < \log m \text{ for all } j = 1,\cdots,n\}$. (Here we choose $\log m$ just to make $\frac{\log m}{\sqrt m} \to 0$). Now, given a local frame $\{e_j(z)\}_{j=1}^n$ of $T^{*(0,1)}_z \tilde D$, then

$$\{e^J(z) : J \text{ is a strictly increasing index with} |J| = q\}$$

is the induced local frame of $T^{*(0,q)}_z \tilde D$. Take the scaled Hermitian metric $\langle\cdot|\cdot\rangle_{F^*_m}$ on $F^*_m T^{*(0,q)}\tilde D$ over $\tilde D_{\log m}$ such that $e^J\left(\frac{z}{\sqrt m}\right)$ forms an orthonormal frame. So for the scaled bundle $F^*_m T^{*(0,q)}\tilde D$ over $\tilde D_{\log m}$, its fiber is locally trivialized by

$$F^*_m T^{*(0,q)}\tilde D\Big|_z := \left\{ \sum_{|J|=q}{}' a_J e^J\left(\frac{z}{\sqrt m}\right) : a_J \in \mathbb{C}\right\}.$$

On the other hand, for $f \in \Omega^{(0,q)}\tilde{D}$, $f = \sum_{|J|=q}{}' f_J(z)e^J(z)$, we define the scaled form

$$F_m^* f(z) := \sum_{|J|=q}{}' f_J\left(\frac{z}{\sqrt{m}}\right) e^J\left(\frac{z}{\sqrt{m}}\right),\ z \in \tilde{D}_{\log m}$$

on $F_m^* T^{*(0,q)}\tilde{D}$ over $\tilde{D}_{\log m}$.

Let $P := \sum_{j=1}^{2n} a_j(z)\frac{\partial}{\partial x_j}$ be an arbitrary partial differential operator of order 1 on $\mathrm{Im}F_m$, and we use the notation $P_m := \sum_{j=1}^{2n} a_J(\frac{z}{\sqrt{m}})\frac{\partial}{\partial x_j}$ to denote the scaled operator on $\tilde{D}_{\log m}$. Under this convention, if we write

$$u = \sum_{|J|=q}{}' u_J e^J$$

then

$$\bar{\partial}_m(F_m^* u) = \frac{1}{\sqrt{m}} F_m^*(\bar{\partial} u).$$

For the weighted L^2 inner product

$$(f|g)_{2mF_m^*\phi} := \int_{\tilde{D}_{\log m}} \langle f|g\rangle_{F_m^*} e^{-2m\phi\left(\frac{z}{\sqrt{m}}\right)} \lambda\left(\frac{z}{\sqrt{m}}\right) dv(z)$$

on the space of compactly supported smooth sections on $F_m^* T^{*(0,q)}\tilde{D}$ over $\tilde{D}_{\log m}$, the formal adjoint with respect to such inner product also satisfies

$$\bar{\partial}_m^*(F_m^* u) = \frac{1}{\sqrt{m}} F_m^*(\bar{\partial}^{*,2m\phi} u).$$

In conclusion, we have

$$\Box_m^{(q)}(F_m^* u) = \frac{1}{m} F_m^*(\Box_{2m\phi}^{(q)} u),$$

where

$$\Box_m^{(q)} := \bar{\partial}_m \bar{\partial}_m^* + \bar{\partial}_m^* \bar{\partial}_m$$

and

$$\Box_{2m\phi}^{(q)} := \bar{\partial}\bar{\partial}^{*,2m\phi} + \bar{\partial}^{*,2m\phi}\bar{\partial}.$$

We are ready to proceed the semi-classical approximation:

Theorem 3.3. *There is $\Box_m^{(q)} = \Box_{2\phi_0}^{(q)} + \epsilon_m P_m$ on $\tilde{D}_{\log m}$, where P_m is a second order partial differential operator and all the coefficients of P_m are uniformly bounded with respect to m in $C^k(\tilde{D}_{\log m})$ for every $k \in \mathbb{N}$, and $\epsilon_m \to 0$ as $m \to \infty$*

Proof. We shall compare $\Box_{2m\phi}^{(q)}$ and $\Box_{2\phi_0}^{(q)}$.

We start with the case of $q = 0$, note that

$$\lambda(z) = 1 + O(|z|),$$

$$\partial_x^\alpha \lambda(z) = c + O(|z|) \text{ for some constant } c,$$

and

$$\langle d\bar{z}_j | d\bar{z}_k\rangle = \delta_{jk} + O(|z|).$$

Then

$$\begin{aligned}
(u|\bar{\partial}^{*,2m\phi}v)_{2m\phi} &= (\bar{\partial}u|v)_{2m\phi} \\
&= \left(\sum_{j=1}^{n}\frac{\partial u}{\partial \bar{z}_j}d\bar{z}_j\middle|\sum_{k=1}^{n}v_k d\bar{z}_k\right)_{2m\phi} \\
&= \sum_{j,k=1}^{n}\left(\frac{\partial u}{\partial \bar{z}_j}d\bar{z}_j\middle|v_k d\bar{z}_k\right)_{2m\phi} \\
&= \sum_{j,k=1}^{n}\int_{\tilde{D}}\frac{\partial u}{\partial \bar{z}_j}\bar{v}_k\langle d\bar{z}_j|d\bar{z}_k\rangle\lambda(z)e^{2m\phi(z)}dv(z) \\
&= -\sum_{j,k=1}^{n}\int_{\tilde{D}}u\frac{\partial}{\partial \bar{z}_j}\left(\bar{v}_k\delta_{jk}\lambda(z)e^{2m\phi(z)}\right)dv(z).
\end{aligned}$$

So we have

$$\bar{\partial}^{*,2m\phi}v = \sum_{j,k=1}^{n}\left(\bar{v}_k\left(\delta_{j,k}+O(|z|)\right)2m\frac{\partial\phi(z)}{\partial\bar{z}_j}-\frac{\partial v_k}{\partial z_j}\left(\delta_{j,k}+O(|z|)\right)-\big(O(1)+O(|z|)\big)\bar{z}_k\right),$$

and hence

$$\begin{aligned}
\Box_{2m\phi}(F_m^* f) = &\sum_{k=1}^{n}\left(\frac{-\partial^2 f}{\partial z_k\partial\bar{z}_k}\left(\frac{z}{\sqrt{m}}\right)+2m\frac{\partial\phi}{\partial z_k}\left(\frac{z}{\sqrt{m}}\right)\frac{\partial f}{\partial\bar{z}_k}\left(\frac{z}{\sqrt{m}}\right)\right) \\
&+\sum_{j,k=1}^{n}\left(\left(2m\frac{\partial\phi}{\partial z_j}\left(\frac{z}{\sqrt{m}}\right)-O(1)-O\left(\frac{|z|}{\sqrt{m}}\right)\right)\frac{\partial f}{\partial\bar{z}_k}\left(\frac{z}{\sqrt{m}}\right)-\frac{\partial^2 f}{\partial z_j\partial\bar{z}_k}\left(\frac{z}{\sqrt{m}}\right)O\left(\frac{|z|}{\sqrt{m}}\right)\right).
\end{aligned}$$

Also,

$$\Box_{2\phi_0}(F_m^* f) = \sum_{k=1}^{n}\frac{-\partial^2 f}{\partial z_k\partial\bar{z}_k}\left(\frac{z}{\sqrt{m}}\right)+2\lambda_k\frac{\bar{z}_k}{\sqrt{m}}\frac{\partial f}{\partial\bar{z}_k}\left(\frac{z}{\sqrt{m}}\right).$$

With the fact that

$$\lim_{m\to\infty}\sup_{z\in\tilde{D}}\left|\partial_x^\alpha\left(2m\phi\left(\frac{z}{\sqrt{m}}\right)-2\Phi_0(z)\right)\right|\to 0,$$

the case for $q=0$ is done.

Now, for general $q\geq 1$, the argument is almost the same, and there is just some slight difference: we have to replace $\delta_{j,k}$ by

$$\epsilon_K^{j,J} := \begin{cases} 0: \{j,j_1,\cdots,j_q\}\neq\{k_1,\cdots,k_{q+1}\} \\ 1: \{j,j_1,\cdots,j_q\}=\{k_1,\cdots,k_{q+1}\}\end{cases},$$

then

$$
\begin{aligned}
(u|\bar{\partial}^{*,2m\phi}v)_{2m\phi} &= (\bar{\partial}u|v)_{2m\phi} \\
&= \left(\sum_{\substack{|J|=q \\ 1\le j\le n}}{}' \frac{\partial u_J}{\partial \bar{z}_j} d\bar{z}_j \wedge d\bar{z}^J \middle| \sum_{|K|=q+1}{}' v_K d\bar{z}^K \right)_{2m\phi} \\
&= \sum_{\substack{|J|=q \\ |K|=q+1 \\ 1\le j\le n}}{}' \int_{\tilde{D}} \frac{\partial u_J}{\partial \bar{z}_j} \bar{v}_K \left\langle d\bar{z}_j \wedge d\bar{z}^J \middle| d\bar{z}^K \right\rangle \lambda(z) dv(z) \\
&= \sum_{\substack{|J|=q \\ |K|=q+1 \\ 1\le j\le n}}{}' \int_{\tilde{D}} \frac{\partial u_J}{\partial \bar{z}_j} \bar{v}_K \epsilon_K^{j,J} \lambda(z) dv(z)
\end{aligned}
$$

the rest calculation is almost the same. $\square$

Corollary 3.2 (Semi-classical elliptic estimate)**.** *For $m \gg 1$, every $r > 0$ such that $\tilde{D}_{2r} \subset \tilde{D}_{\log m}$ and $s \in \mathbb{N}$, then there exists a constant $C_{r,s} > 0$ independent of m and the point x_0 satisfying*

$$
\|u\|_{2mF_m^*\phi, H^{s+2}, \tilde{D}_r} \le C_{s,r} \left(\left\| \Box_m^{(q)} u \right\|_{2mF_m^*\phi, L^2, \tilde{D}_{2r}} + \|u\|_{2mF_m^*\phi, H^s, \tilde{D}_{2r}} \right)
$$

for any $u \in F_m^*\Omega^{(0,q)}(\tilde{D}_{\log m})$.

Proof. Note that on the compact set $\tilde{D}_r$,

$$
e^{-2m\phi\left(\frac{z}{\sqrt{m}}\right)} = e^{-2\Phi_0(z) - O(\frac{|z|^3}{\sqrt{m}})}
$$

and

$$
\lambda\left(\frac{z}{\sqrt{m}}\right) = 1 + O\left(\frac{|z|}{\sqrt{m}}\right)
$$

are both bounded, so apply the elliptic estimate to $\Box_{2\phi_0}^{(q)}$, we have

$$
\begin{aligned}
\|u\|_{2mF_m^*\phi, H^{s+2}, \tilde{D}_r} &:= \sum_{\substack{|\alpha|\le s+2 \\ |J|=q}}{}' \int_{\tilde{D}_r} |\partial_x^\alpha u|^2 e^{-2m\phi\left(\frac{z}{\sqrt{m}}\right)} \lambda\left(\frac{z}{\sqrt{m}}\right) dv(z) \\
&\le C_{x_0,r,1} \sum_{\substack{|\alpha|\le s+2 \\ |J|=q}}{}' \int_{\tilde{D}_r} |\partial_x^\alpha u|^2 dv(z) \\
&\le C_{x_0,r,s,2} \left(\left\| \Box_{2\phi_0}^{(q)} u \right\|_{H^s, \tilde{D}_{2r}} + \|u\|_{H^s, \tilde{D}_{2r}} \right).
\end{aligned}
$$

By **Theorem 3.3**, for $m \gg 1$,

$$\begin{aligned}\left\|\Box_{2\phi_0}^{(q)}u\right\|_{H^s} &= \left\|\Box_m^{(q)}u-\epsilon_m P_m u\right\|_{H^s,\tilde{D}_{2r}}\\ &\le \left\|\Box_m^{(q)}u\right\|_{H^s,\tilde{D}_{2r}}+|\epsilon_m|\cdot\|P_m u\|_{H^s,\tilde{D}_{2r}}\\ &\le \left\|\Box_m^{(q)}u\right\|_{H^s,\tilde{D}_{2r}}+C\|u\|_{H^{s+2},\tilde{D}_{2r}}\\ &\le \left\|\Box_m^{(q)}u\right\|_{H^s,\tilde{D}_{2r}}+C'\|u\|_{2mF_m^*\phi,H^{s+2},\tilde{D}_r}\text{ for some constant }0<C'\ll 1.\end{aligned}$$

Plug this result into the above estimate, after some arrangement we can get

$$\begin{aligned}\|u\|_{2mF_m^*\phi,s+2,\tilde{D}_r} &\le C_{x_0,r,s,3}\left(\left\|\Box_m^{(q)}u\right\|_{H^s,\tilde{D}_{2r}}+\|u\|_{H^s,\tilde{D}_{2r}}\right)\\ &\le C_{x_0,r,s,4}\left(\left\|\Box_m^{(q)}u\right\|_{2mF_m^*\phi,H^s,\tilde{D}_{2r}}+\|u\|_{2mF_m^*\phi,H^s,\tilde{D}_{2r}}\right).\end{aligned}$$

Finally, by compactness of X, after taking finite cover $\tilde{D}_r$, we can conclude the constant is independent of x_0. □

Lemma 3.1. *For all $u \in \Omega_m^{(0,q)}(X)$, on any canonical coordinate patch D we have*

$$\Box_{b,m}^{(q)}u = e^{im\theta}e^{-m\phi}\Box_{2m\phi}^{(q)}(e^{m\phi}e^{-im\theta}u).$$

Proof. We claim that

$$\bar{\partial}_b u = e^{im\theta}e^{m\phi}\bar{\partial}(e^{m\phi}e^{-im\theta}u)$$

and

$$\bar{\partial}_b^* u = e^{im\theta}e^{m\phi}\bar{\partial}^{*,2m\phi}(e^{m\phi}e^{-im\theta}u).$$

The first one is some how easier. Let $u \in \Omega_m^{(0,q)}(D)$, where D is the canonical patch, then we can write

$$u = \sum_{|J|=q}{}' u_J d\bar{z}^J = \tilde{u}(z)e^{im\theta}.$$

We benefits from the **Theorem 3.1** and write

$$\begin{aligned}\bar{\partial}_b u &= \sum_{\substack{|J|=q\\1\le j\le n}}{}'\left(\frac{\partial u_J}{\partial\bar{z}_j}-i\frac{\partial\phi(z)}{\partial\bar{z}_j}\frac{\partial u_J}{\partial\theta}\right)d\bar{z}_j\wedge d\bar{z}^J\\ &= e^{im\theta}\sum_{\substack{|J|=q\\1\le j\le n}}{}'\left(\frac{\partial\tilde{u}_J}{\partial\bar{z}_j}+m\frac{\partial\phi(z)}{\partial\bar{z}_j}\frac{\partial\tilde{u}_J}{\partial\theta}\right)d\bar{z}_j\wedge d\bar{z}^J\\ &= e^{im\theta}e^{-m\phi}\bar{\partial}(e^{m\phi}e^{-im\theta}u).\end{aligned}$$

To avoid the boundary term of the integration by parts, consider the cut-off function

$$\chi:\begin{cases}\chi(\theta)\in\mathscr{C}_0^\infty(-\delta,\delta)\\ \int_{\mathbb{R}}\chi(\theta)d\theta=1\end{cases}.$$

Now, since $\bar{\partial}_b^* u \in \Omega_m^{(0,q-1)}X$, write

$$\bar{\partial}_b^* u = \tilde{v}(z)e^{im\theta}.$$

Take the pairing with respect to $g(z) \in \Omega_0^{(0,q)}\tilde{D}$ there is

$$\begin{aligned}\left(\bar{\partial}_b^* u \middle| e^{im\theta}\chi(\theta)g(z)e^{-2m\phi(z)}\right) &= \left(e^{im\theta}\tilde{v}(z)\middle| e^{im\theta}\chi(\theta)g(z)e^{-2m\phi(z)}\right)\\ &= \left(\tilde{v}(z)\middle| g(z)\right)_{2m\phi}.\end{aligned}$$

On the other hand,

$$\left(\bar{\partial}_b^* u \middle| e^{im\theta}\chi(\theta)g(z)e^{-2m\phi(z)}\right) = \left(u\middle|\bar{\partial}_b\left(e^{im\theta}\chi(\theta)g(z)e^{-2m\phi(z)}\right)\right).$$

Follow the calculation earlier,

$$\bar{\partial}_b\left(e^{im\theta}\chi(\theta)g(z)e^{-2m\phi(z)}\right)$$

is

$$\sum_{\substack{|J|=q-1\\ 1\le j\le n}}{}' \left(\chi(\theta)e^{im\theta}\frac{\partial\left(e^{-2m\phi(z)}g_J(z)\right)}{\partial\bar{z}_j} + ie^{-2m\phi(z)}g_J(z)\frac{\partial\phi}{\partial\bar{z}_j}\frac{\partial\left(e^{im\theta}\chi(\theta)\right)}{\partial\theta}\right)d\bar{z}_j\wedge d\bar{z}^J$$

which can be arranged to

$$\chi(\theta)e^{im\theta}\bar{\partial}\left(e^{-2m\phi(z)}g(z)\right) + e^{-2m\phi(z)}e^{im\theta}\left(i\chi'(\theta) - m\chi(\theta)\right)\bar{\partial}\phi\wedge g.$$

With the help of

$$\int_{\mathbb{R}}\chi'(\theta)d\theta = 0$$

and

$$\int_{\mathbb{R}}\chi(\theta)d\theta = 1,$$

we can rewrite

$$\begin{aligned}\left(u\middle| e^{im\theta}e^{-im\phi(z)}\bar{\partial}\left(e^{-m\phi(z)}g(z)\chi(\theta)\right)\right) &= \left(u\middle| e^{im\theta}\chi(\theta)\left(\bar{\partial}(e^{-2m\phi(z)}g(z)) - me^{-2m\phi(z)}\bar{\partial}\phi\wedge g\right)\right)\\ &= \left(u\middle| e^{im\theta}\chi(\theta)e^{-m\phi(z)}\bar{\partial}\left(e^{-m\phi(z)}g(z)\right)\right)\\ &= \left(\tilde{u}(z)e^{m\phi(z)}\middle|\bar{\partial}\left(e^{-m\phi(z)}g(z)\right)\right)_{2m\phi}\\ &= \left(e^{-m\phi(z)}\bar{\partial}^{*,2m\phi}\left(\tilde{u}(z)e^{m\phi(z)}\right)\middle| g(z)\right)_{2m\phi}.\end{aligned}$$

So $\bar{\partial}_b^* u = \tilde{v}(z)e^{im\theta} = e^{im\theta}e^{-m\phi(z)}\bar{\partial}^{*,2m\phi}\left(\tilde{u}(z)e^{m\phi(z)}\right)$, as desired.

□

3.4. **The model case.** To understand the asymptotic behavior of the m-th Szegö kernel, we look for its close cousin Bergman kernel on the model case $\mathbb{C}^n$. The idea is given by the submean estimate of eigenvalues appeared in Berman [2]. We begin with some basic calculations:

Lemma 3.2.

For $|J| = q$, then we have

$$\Box_{2\phi_0}^{(q)}(fd\bar{z}^J) = \left(\sum_{j\in J}\frac{\partial}{\partial\bar{z}_j}\frac{\partial}{\partial\bar{z}_j}^{*,2\phi_0} + \sum_{j\notin J}\frac{\partial}{\partial\bar{z}_j}^{*,2\phi_0}\frac{\partial}{\partial\bar{z}_j}\right)fd\bar{z}^J.$$

Proof. First recall that

$$\begin{aligned}\Box^{(q)}_{2\phi_0} &:= \bar\partial^{*,2\phi_0}\bar\partial + \bar\partial\bar\partial^{*,2\phi_0}\\ &= \sum_{i,j=1}^{n}\left(\frac{\partial}{\partial\bar z_i}^{*,2\phi_0}\frac{\partial}{\partial\bar z_j}d\bar z_i^{*}d\bar z_j + \frac{\partial}{\partial\bar z_i}\frac{\partial}{\partial\bar z_j}^{*,2\phi_0}d\bar z_i d\bar z_j^{*}\right)\end{aligned}$$

where

$$d\bar z_i(d\bar z^I) := d\bar z_i\wedge d\bar z^I$$

and

$$d\bar z_i^{*}(d\bar z^I) := d\bar z_i \lrcorner d\bar z^I.$$

Observe that we have the relation

$$d\bar z_i^{*}d\bar z_j + d\bar z_j^{*}d\bar z_i = 0,\ i\neq j,$$

$$d\bar z_i^{*}d\bar z_i(d\bar z^I) = \begin{cases} d\bar z^I, \text{ if } i\notin I\\ 0, \text{ if } i\in I\end{cases},$$

and

$$d\bar z_i d\bar z_i^{*}(d\bar z^I) = \begin{cases} 0, \text{ if } i\notin I\\ d\bar z^I, \text{ if } i\in I\end{cases}.$$

Hence, for $|I| = q$,

$$\Box^{(q)}_{2\phi_0}(fd\bar z^J) = \left(\sum_{j\in J}\frac{\partial}{\partial\bar z_j}\frac{\partial}{\partial\bar z_j}^{*,2\phi_0} + \sum_{j\notin J}\frac{\partial}{\partial\bar z_j}^{*,2\phi_0}\frac{\partial}{\partial\bar z_j}\right)fd\bar z^J.$$

$\square$

Lemma 3.3. *Write $u = \sum'_{|J|=q}u_J d\bar z^J$. If*

$$\Box^{(q)}_{2\phi_0}u = 0,$$

then

$$\frac{\partial}{\partial\bar z_j}u_J = 0 \textit{ if } j\notin J$$

and

$$\frac{\partial}{\partial\bar z_j}^{*,2\phi_0}u_J = 0 \textit{ if } j\in J.$$

Proof. Since $\mathbb{C}^n$ is non-compact, to eliminate the boundary term in the calculation via integration by parts, we construct the cut off function $\chi(z)\in\mathscr{C}_0^\infty(\mathbb{C}^n,\mathbb{R})$ such that $\chi = 1$ near $z = 0$, and take $\chi_R(z) := \chi(\frac{z}{R})$.

By the assumption and **Lemma 3.2**, we have

$$\begin{aligned}0 &= \left(\left(\sum_{j\in J}\frac{\partial}{\partial\bar z_j}\frac{\partial}{\partial\bar z_j}^{*,2\phi_0} + \sum_{j\notin J}\frac{\partial}{\partial\bar z_j}^{*,2\phi_0}\frac{\partial}{\partial\bar z_j}\right)u_J\,\middle|\,\chi_R^2 u_J\right)_{2\phi_0}\\ &= \sum_{j\in J}\left(\frac{\partial}{\partial\bar z_j}^{*,2\phi_0}u_J\,\middle|\,\frac{\partial}{\partial\bar z_j}^{*,2\phi_0}(\chi_R^2u_J)\right)_{2\phi_0} + \sum_{j\notin J}\left(\frac{\partial}{\partial\bar z_j}u_J\,\middle|\,\frac{\partial}{\partial\bar z_j}(\chi_R^2u_J)\right)_{2\phi_0}.\end{aligned}$$

For $\epsilon \in (0,1)$, each component in the second term has the lower bound

$$\left(\frac{\partial u_J}{\partial \bar{z}_j}\middle|\frac{\partial \chi}{\partial \bar{z}_j}\frac{2\chi_R u_J}{R}\right)_{2\phi_0} + \left\|\chi_R\frac{\partial u_J}{\partial \bar{z}_j}\right\|^2_{2\phi_0,L^2} = 2\left(\chi_R\frac{\partial u_J}{\partial \bar{z}_j}\middle|\frac{\partial \chi}{\partial \bar{z}_j}\frac{u_J}{R}\right)_{2\phi_0} + \left\|\chi_R\frac{\partial u_J}{\partial \bar{z}_j}\right\|^2_{2\phi_0,L^2}$$

$$\geq -2\left\|\chi_R\frac{\partial u_J}{\partial \bar{z}_j}\right\|_{2\phi_0,L^2}\frac{1}{R}\left\|\frac{\partial \chi}{\partial \bar{z}_j}u_J\right\|_{2\phi_0,L^2} + \left\|\chi_R\frac{\partial u_J}{\partial \bar{z}_j}\right\|^2_{2\phi_0,L^2}$$

$$\geq -\epsilon\left\|\chi_R\frac{\partial u_J}{\partial \bar{z}_j}\right\|^2_{2\phi_0,L^2} - \frac{1}{\epsilon}\frac{C}{R^2} + \left\|\chi_R\frac{\partial u_J}{\partial \bar{z}_j}\right\|^2_{2\phi_0,L^2}$$

(the constant C comes from the compact support of χ)

Take $R \to \infty$ then we find that R.H.S converges to $(1-\epsilon)\left\|\frac{\partial u}{\partial \bar{z}_j}\right\|^2_{2\phi_0,,L^2}$, and combine the same calculus for the first term, we then have so we have

$$\sum_{j\in J}\left\|\frac{\partial u}{\partial \bar{z}_j}^{*,2\phi_0}\right\|^2_{2\phi_0,L^2} + \sum_{j\notin J}\left\|\frac{\partial u_J}{\partial \bar{z}_j}\right\|^2_{2\phi_0,L^2} = 0$$

as desired. □

In conclusion, the above two observations suggest that if we put the extremal function in the direction e^J on $\mathbb{C}^n$ by

$$S^{(q)}_{J,\mathbb{C}^n}(0) := \sup\left\{|u_J(0)|^2 : u \in \Omega^{(0,q)}(\mathbb{C}^n),\ \Box^{(q)}_{2\phi_0}u = 0,\ \|u\|_{2\phi_0,L^2} = 1\right\},$$

where $\phi_0 := \sum_{j=1}^n \lambda_j|z_j|^2$, then

Theorem 3.4.

$$\sum_{|J|=q}{}' S^{(q)}_{J,\mathbb{C}^n}(0) = \begin{cases}(\pi)^{-n}|\lambda_1\cdots\lambda_n| \textit{ if exact } q \textit{ of } \lambda_j \textit{ are negative and } n-q \textit{ of } \lambda_j \textit{ are positive}\\ 0: \textit{ otherwise}\end{cases}.$$

Proof. We start from the case $q = 0$. Note that by **Lemma 3.3**, we know that for $u \in \mathscr{C}^\infty(\mathbb{C}^n)$, $\Box_{2\phi_0}u = 0$, then u is holomorphic, i.e. $|u|^2$ is subharmonic. So by submean inequality in terms of polar coordinate

$$|u(0)|^2 \leq (2\pi)^{-n}\int_{\theta\in[0,2\pi)} u(re^{i\theta})d\theta,\ \text{for any } r > 0.$$

where the notation $\{\theta \in [0,2\pi)\} := \{\theta_j \in [0,2\pi),\ j = 1,\cdots,n\}$, $re^{i\theta} := (r_1e^{\theta_1},\cdots,r_ne^{\theta_n})$, $r := (\sum_{j=1}^n r_j^2)^{\frac{1}{2}}$, $d\theta := d\theta_1\cdots d\theta_n$. Consider integration with respect to r by

$$\int_{r=0}^\infty |u(0)|^2 r_1\cdots .r_n e^{-\sum_{j=1}^n 2\lambda_j {r_j}^2}dr \leq (2\pi)^{-n}\int_{r=0}^\infty\int_{\theta\in[0,2\pi)} u(re^{i\theta})e^{-\sum_{j=1}^n 2\lambda_j {r_j}^2}r_1\cdots r_n drd\theta.$$

Note that

$$\text{R.H.S} = (2\pi)^{-n}\int_{\mathbb{C}^n} u(z)e^{-2\Phi_0(z)}2^{-n}dv(z) = \frac{1}{(4\pi)^n}\|u\|^2_{2\phi_0,L^2} = \frac{1}{(4\pi)^n}$$

and

$$\text{L.H.S converges to } |u(0)|^2\frac{1}{4\lambda_1\cdots,\lambda_n} \text{ if } \lambda_j > 0 \text{ for all } j.$$

So

$$|u(0)|^2 \leq \frac{1}{\pi^n}\lambda_1\cdots\lambda_n \text{ if } \lambda_j > 0 \text{ for all } j$$

and

$$|u(0)|^2 = 0 \text{ if } \lambda_j < 0 \text{ for any } j.$$

On the other hand, if $\lambda_j > 0$ for all j, the constant function $f := \frac{1}{\pi^n}\lambda_1 \cdots \lambda_n$ satisfies $f \in \mathscr{C}^\infty(\mathbb{C}^n)$, $\Box_{2\phi_0} f = 0$ and $\|f\|_{2\phi_0} = 1$. Hence $S_{\mathbb{C}^n}(0) = \frac{1}{\pi^n}\lambda_1 \cdots \lambda_n$.

Now, for any $q \geq 1$, we manage to deduce the result from the case $q = 0$. To begin with, we calculate $\frac{\partial}{\partial \bar{z}_j}^{*,2\phi_0}$ by taking paring with respect to test functions f, g:

$$\begin{aligned}
\left(\frac{\partial}{\partial \bar{z}_j}^{*,2\phi_0} f \middle| g \right)_{2\phi_0} &= \left(f \middle| \frac{\partial}{\partial \bar{z}_j} g \right)_{2\phi_0} \\
&= \int_{\mathbb{C}^n} f \overline{\frac{\partial g}{\partial \bar{z}_j}} e^{-2\phi_0(z)} dv(z) \\
&= \int_{\mathbb{C}^n} f e^{-2\phi_0(z)} \frac{\partial \bar{g}}{\partial z_j} dv(z) \\
&= -\int_{\mathbb{C}^n} \frac{\partial}{\partial z_j} \left(f e^{-2\phi_0(z)} \right) \bar{g} dv(z)
\end{aligned}$$

So we get $\frac{\partial}{\partial \bar{z}_j}^{*,2\phi_0} f = -\frac{\partial}{\partial z_j}\left(f e^{-2\phi_0(z)} \right)$, i.e.

$$\frac{\partial}{\partial \bar{z}_j}^{*,2\phi_0} = -\frac{\partial}{\partial z_j} + 2\lambda_j \bar{z}_j.$$

Apply **Lemma 3.3**, we immediately have

$$\frac{\partial}{\partial \bar{z}_j} u_J = 0 \text{ if } j \notin J$$

and

$$\frac{\partial}{\partial \bar{z}_j}^{*} u_J = \left(-\frac{\partial}{\partial z_j} + 2\lambda_j \bar{z}_j \right) u_J = 0 \text{ if } j \in J.$$

Now, consider the function

$$\tilde{u}_J(z) := e^{-2\sum_{j\in J} \lambda_j |z_j|^2} u_J(\zeta)$$

where

$$\zeta_j := \bar{z}_j \text{ if } j \in J$$

and

$$\zeta_j := z_j \text{ if } j \notin J.$$

Then

$$\frac{\partial}{\partial \bar{z}_k} \tilde{u}_J = \frac{\partial}{\partial \bar{z}_k} \left(e^{-2\sum_{j\in J} \lambda_j |z_j|^2} u_J(\zeta) \right)$$

is either

$$e^{-2\sum_{j\in J} \lambda_j |z_j|^2} \left(\frac{\partial}{\partial \bar{z}_k} u_J(\zeta) \right) = 0 \text{ if } k \notin J$$

or

$$e^{-2\sum_{j\in J} \lambda_j |z_j|^2} \left(-2\lambda_k z_k + \frac{\partial}{\partial \bar{z}_k} u_J(\zeta) \right) = 0 \text{ if } k \in J.$$

This means $\tilde{u}_J$ is holomorphic; moreover, we have

$$|u_J|_{2\phi_0} = |\tilde{u}_J|_{2\phi_{0,J}}$$

where

$$\phi_{0,J} := -\sum_{j\in J}\lambda_j|z_j|^2 + \sum_{j\notin J}\lambda_j|z_j|^2$$

and

$$u_J(0) = \tilde{u}_J(0).$$

If we are not in the case that exactly q of λ_j are negative, we can rewrite

$$\phi_{0,J} := \sum_{j=1}^{n}\tilde{\lambda}_j|z_j|^2$$

where at least one of the $\tilde{\lambda}_j < 0$. Then by the assumption $|u|_{2\phi_0,L^2} = 1$ and the Fubini-Tonelli's theorem, we have

$$\int_{\mathbb{C}^n}|\tilde{u}_J(0,\cdots,z_j,\cdots 0)|^2 e^{-\tilde{\lambda}_j|z_j|^2}dv(z) < \infty.$$

Apply the submean inequality as before, we acquire

$$\tilde{u}_J(0) = 0,$$

i.e.

$$u_J(0) = 0.$$

On the other hand, if we have exactly q of λ_j are negative, we may assume

$$\lambda_1,\cdots,\lambda_q < 0.$$

The argument just used implies

$$u_J(z) \equiv 0 \text{ if } J \neq \{\lambda_1,\cdots,\lambda_q\}$$

and

$$1 = \|u\|^2_{2\phi_0,L^2} = \sum_{|J=q|}{}'\|u_J\|^2_{2\phi_0,L^2} = \left\|u_{\{\lambda_1,\cdots,\lambda_q\}}\right\|^2_{2\phi_0,L^2} = \left\|\tilde{u}_{\{\lambda_1,\cdots,\lambda_q\}}\right\|^2_{2\phi_{0,\{\lambda_1,\cdots,\lambda_q\}},L^2}.$$

Again, by using submean inequality, there is

$$\begin{aligned}\left|u_{\{\lambda_1,\cdots,\lambda_q\}}(0)\right|^2 &= \left|\tilde{u}_{\{\lambda_1,\cdots,\lambda_q\}}(0)\right|^2 \\ &\leq (\pi)^{-n}(-\lambda_1)\cdots(-\lambda_q)(\lambda_{q+1})\cdots(\lambda_n) \\ &= (\pi)^{-n}|\lambda_1|\cdots|\lambda_n|.\end{aligned}$$

This is exactly the case when $q = 0$, so we can deduce that

$$\sum_{|J|=q}{}' S^{(q)}_{J,\mathbb{C}^n}(0) = (\pi)^{-n}|\lambda_1\cdots\lambda_n|.$$

□

3.5. **The local S^1-equivariant weak CR Morse inequality.** Inspired by the result in the last section, we can study the the local S^1-equivariant weak CR Morse inequality by relating the Szegö kernel and the extremal function:

Lemma 3.4.

(1) *Given any orthonormal basis $\{f_k\}_{k=1}^{d_m}$ of $\ker\square_{b,m}^{(q)}$, consider*

$$\Pi_m^{(q)}(x) := \sum_{k=1}^{d_m} |f_k(x)|^2$$

then such sum is independent of the choice of the orthonormal basis.

(2) *For every orthonormal frame $\{e^J : |J| = q, J \text{ is strictly incrasing}\}$ at $T_x^{*(0,q)}X$, locally write the $u \in \Omega^{(0,q)}(X)$ as $u = \sum'_{|J|=q} u_J e^J$, then the extremal function in the direction e^J is defined by*

$$S_{m,J}^{(q)}(y) := \left\{\sup |u_J(y)|^2 : u \in \ker\square_{b,m}^{(q)}, \|u\|_{L^2}^2 := \int_X |u|^2 dV_X = 1\right\}$$

And we have the relation

$$\Pi_m^{(q)}(x) = \sum_{|J|=q}{}' S_{m,J}^{(q)}(x) \text{ for all } x \in X.$$

Proof.

(1) For two orthonormal basis $\{f_i\}$ and $\{g_j\}$ for $\ker\square_{b,m}^{(q)}$, write $g_j = a_j^i f_i$, then

$$\begin{aligned}\delta_{ij} &= (g_i|g_j)\\ &= a_i^k \bar{a}_j^l (f_k|f_l)\\ &= a_i^k \bar{a}_j^l \delta_{kl}\\ &= \sum_{k=1}^{d_m} a_i^k \bar{a}_j^k.\end{aligned}$$

This is equivalent to

$$\sum_{k=1}^{d_m} a_k^i \bar{a}_k^j = \delta_{ij}.$$

Hence $\sum_{k=1}^{d_m} |g_k|^2 = \sum_{k=1}^{d_m} \langle g_k|g_k\rangle = \sum_{k=1}^{d_m} a_k^i \bar{a}_k^j \langle f_i|f_j\rangle = \sum_{k=1}^{d_m} \delta_{ij}\langle f_i|f_j\rangle = \sum_{i=1}^{d_m} |f_i|^2$.

(2) First, for any $\alpha \in \ker\square_{b,m}^{(q)}$ with $\|\alpha\|_{L^2} = 1$, then α must be contained in some orthonormal basis for $\ker\square_{b,m}^{(q)}$, say $\{f_k\}_{k=1}^{d_m}$. Decompose

$$\Pi_m^{(q)}(x) = \sum_{|J|=q}{}'\Pi_{m,J}^{(q)}(x) := \sum_{|J|=q}{}' \sum_{k=1}^{d_m} |f_{k,J}(x)|^2.$$

Note that $|\alpha_J|^2 \le \Pi_{m,J}^{(q)}(x)$, so $\Pi_m^{(q)}(x) \ge \sum_{|J|=q}{}' S_{m,J}^{(q)}(x)$ for all $x \in X$.

Conversely, fix any $x_0 \in X$, and let $\{f_k\}_{k=1}^{d_m}$ be an orthonormal basis for $\ker\square_{b,m}^{(q)}$, then for any x near X, construct

$$\beta(x) := \left(\sum_{k=1}^{d_m} |f_{k,J}(x_0)|^2\right)^{-\frac{1}{2}} \sum_{k=1}^{d_m} \bar{f}_{k,J}(x_0) f_k(x).$$

Clearly, we have $\Box^{(q)}_{b,m}\beta = 0$, and also

$$\begin{aligned}\|\beta\|^2_{L^2} &= \int_X \frac{1}{\sum_{k=1}^{d_m} |f_{k,J}(x_0)|^2} \sum_{k,l=1}^{d_m} \left\langle \bar{f}_{k,J}(x_0) f_k(x) \middle| \bar{f}_{l,J}(x_0) f_l(x) \right\rangle dV_X(x) \\ &= \frac{\sum_{k=1}^{d_m} |f_{k,J}(x_0)|^2 \|f_k\|^2}{\sum_{k=1}^{d_m} |f_{k,J}(x_0)|^2} \\ &= 1.\end{aligned}$$

Finally, there is $\Pi^{(q)}_{m,J}(x_0) = \sum_{k=1}^{d_m} |f_{k,J}|^2 = |\beta_J(x_0)|^2 \leq S^{(q)}_{m,J}(x_0)$.

□

We are now ready to derive **Theorem 2.1**:

(1) Fix any $x_0 \in X$, take a canonical patch $D = \tilde{D} \times (-\delta, \delta)$ around x_0, and the pairing (z, θ, ϕ) such that it is trivial at x_0.

Now, for any $u \in \ker \Box^{(q)}_{b,m}(X)$, then on D we have

$$u(z,\theta) = \tilde{u}(z) e^{im\theta} \text{ by } Tu = imu,\ T = \frac{\partial}{\partial\theta}$$

and

$$\Box^{(q)}_{2m\phi} v_m(z) = 0 \text{ by } \textbf{Lemma 3.1},\ \text{where } v_m(z) := e^{m\phi(z)}\tilde{u}(z)$$

If we also assume $1 = \|u\|_{L^2} := \int_X |u|^2 dV_X$, then

$$\begin{aligned}1 &\geq \int_D |u|^2 dV_X \\ &= \int_D |v_m(z)|^2 e^{-2m\phi(z)} |e^{2im\theta}| \lambda(z) dv(z) d\theta \\ &= \int_{-\delta}^{\delta} (\int_{\tilde{D}} |v_m(z)|^2 e^{-2m\phi(z)} \lambda(z) dv(z)) d\theta\end{aligned}$$

Consider the scaling $\tilde{v}_m(z) := m^{-\frac{n}{2}} e^{m\phi\left(\frac{z}{\sqrt{m}}\right)} \tilde{u}\left(\frac{z}{\sqrt{m}}\right) = m^{-\frac{n}{2}} F_m^*(e^{m\phi(z)}\tilde{u}(z))$, then the above calculation suggests that

$$\|\tilde{v}_m(z)\|^2_{\tilde{D}_r, 2mF_m^*\phi(z)} = \int_{\tilde{D}_r} |\tilde{v}_m(z)|^2_{F_m^*} e^{-2m\phi(\frac{z}{m})} \lambda\left(\frac{z}{\sqrt{m}}\right) dv(z) \leq \int_{\tilde{D}} |v_m(z)|^2 e^{-2m\phi(z)} \lambda(z) dv(z) \leq \frac{1}{2\delta}$$

for any r such that $\tilde{D}_{2r} \subset \tilde{D}_{\log m}$.

Also, we can find that the scaled Laplacian

$$\Box^{(q)}_m \tilde{v}_m(z) = 0 \text{ by } \Box^{(q)}_{2m\phi} v_m(z) = 0.$$

These observations motivate us to apply the semi-classical elliptic estimate **Proposition 3.2**, so we have

$$\|\tilde{v}_m(z)\|_{\tilde{D}_r, 2mF_m^*\phi, H^{s+2}} \leq C'_{s,r,\delta}.$$

Here, we may assume $C'_{s,r,\delta}$ is independent of x_0, since the compactness of X ensures δ can be picked independent of x_0. Apply the results above and the Sobolev's inequality,

$$m^{-n}|u(x_0)|^2 = m^{-n}|\tilde{u}(z(x_0))|^2 = |\tilde{v}_m(0)|^2 \leq C\|\tilde{v}_m(z)\|_{\tilde{D}_r, 2mF_m^*\phi, H^{s+2}} \leq C'$$

where C' is a constant independent of x_0 and m.

So we can conclude that

$$\sup\{m^{-n}\Pi_m^{(q)}(z) : m \in \mathbb{N}, x \in X\} < \infty.$$

(2) On the other hand, by the definition of lim sup, we can find a sequence $\{u_{m_k}\}$ in $\mathcal{H}^{(q)}_{b,m_k}$ with $\|u_{m_k}\|_{L^2} = 1$ such that

$$\limsup_{m\to\infty} m^{-n} S^{(q)}_{m,J}(x_0) = \lim_{k\to\infty} m_k^{-n} |u_{m_k,J}(x_0)|^2.$$

Again, on D we have

$$u_{m_k} = \tilde{u}_{m_k} e^{im_k\theta}$$

and

$$\Box^{(q)}_{2m\phi} e^{m_k\phi(z)} \tilde{u}_{m_k}(z) = 0 \text{ by } \textbf{Lemma 3.1}$$

Also, let

$$\tilde{v}_{m_k}(z) := m^{-\frac{n}{2}} e^{m_k\phi(\frac{z}{\sqrt{m_k}})} \tilde{u}(\frac{z}{\sqrt{m_k}}) = m_k{}^{-\frac{n}{2}} F^*_{m_k}(e^{m_k\phi(z)}\tilde{u}(z)),$$

$$\|\tilde{v}_{m_k}(z)\|^2_{\tilde{D}_{\log m_k}, 2mF^*_m\phi(z)} \le \frac{1}{2\delta} \text{ by } \|u_{m_k}\|_{L^2} = 1$$

and

$$\Box^{(q)}_m \tilde{v}_{m_k}(z) = 0.$$

Similar to the case in the first part, we apply the semi-classical elliptic estimate **Corollary 3.2**:

For $m_k \gg 1$, every $r > 0$ such that $\tilde{D}_{2r} \subset \tilde{D}_{\log m}$ and $s \in \mathbb{N}$, then there exists a constant $c_{r,s} > 0$ independent of m_k and the point x_0 satisfying

$$\|\tilde{v}_{m_k}\|_{2mF^*_{m_k}\phi, H^{s+2}, \tilde{D}_r} \le C_{s,r}\left(\|\tilde{v}_{m_k}\|_{2m_kF^*_{m_k}\phi, L^2, \tilde{D}_{2r}} + \left\| \Box^{(q)}_m \tilde{v}_{m_k} \right\|_{2m_kF^*_{m_k}\phi, H^s, \tilde{D}_{2r}} \right).$$

This leads to

$$\|\tilde{v}_{m_k}\|_{2mF^*_{m_k}\phi, H^{s+2}, \tilde{D}_r} \le C_{s,r,\delta}.$$

Outside $\tilde{D}_{\log m_k}$, we extend $\tilde{v}_{m_k}$ by zero. Then by Sobolev's compact embedding theorem and diagonal process, we find a subsequence

$$\tilde{v}_{m_{k_j}} \to v := \sum_{|J|=q}{}' v_J(z) d\bar{z}^J \in \Omega^{(0,q)}(\mathbb{C}^n)$$

in $\mathscr{C}^\infty(K)$ topology for any compact set $K \subset \mathbb{C}^n$.

Moreover, the limit process leaves

$$\Box^{(q)}_{2\phi_0} v = 0$$

and

$$\int_{\mathbb{C}^n} |v(z)|^2 e^{-2\phi_0(z)} dv(z) \le \frac{1}{2\delta}.$$

So we obtain

$$|v_J(0)|^2 \le \frac{1}{2\delta} S^{(q)}_{\mathbb{C}^n, J}(0)$$

and

$$\begin{aligned}\limsup_{m\to\infty} m^{-n} S^{(q)}_{m,J}(x_0) &= \lim_{j\to\infty} m_{k_j}^{-n} \left|\left(u_{m_{k_j}}\right)_J(x_0)\right|^2\\ &= \lim_{j\to\infty} \left|\left(\tilde{v}_{m_{k_j}}\right)_J(0)\right|^2\\ &= |v_J(0)|^2\\ &\le \frac{1}{2\delta} S^{(q)}_{\mathbb{C}^n,J}(0).\end{aligned}$$

Finally, combine the above calculation, **Theorem 3.4** and **Theorem 3.4**

$$\begin{aligned}\limsup_{m\to\infty} m^{-n}\Pi^{(q)}_m(x_0) &= \limsup_{m\to\infty} m^{-n} {\sum_{|J|=q}}' S^{(q)}_{m,J}(x_0)\\ &\le {\sum_{|J|=q}}' \limsup_{m\to\infty} m^{-n} S^{(q)}_{m,J}(x_0)\\ &\le \frac{1}{2\delta} S^{(q)}_{\mathbb{C}^n,J}(0)\\ &= \frac{1}{2\delta}\frac{1}{2\pi^n}|\lambda_1\cdots\lambda_n| 1_{X(q)}(x_0).\end{aligned}$$

With the third part in **Theorem 3.1**, we can choose $\delta = \frac{\pi}{k} - \epsilon$ when $x_0 \in X_k$, so we can deduce for all $k \in \mathbb{N}$, $x_0 \in X_k \neq \emptyset$, then for all $q = 0, 1, \cdots, n$

$$\limsup_{m\to\infty} m^{-n}\Pi^{(q)}_m(x_0) \le \frac{k^n}{2\pi^{n+1}}|\det \mathcal{L}_x| 1_{X(q)}(x_0).$$

4. Asymptotic bounds for the dimension of torus equivariant CR sections

4.1. Basic settings and the operator $-iT_0$.

We start with some basic terminology, and illustrate the idea of Hendrick–Hsiao–Li [8] , where they use the transversality condition to reduce our problem to the case for $\mathbb{R}$-action

Let $T^d \curvearrowright X$ be a CR manifold of $\dim_{\mathbb{R}} X = 2n+1$, $n \ge 1$ with a torus action. We denote the group action as

$$(e^{i\theta_1}, \cdots, e^{i\theta_d}) : T^d \times X \to X \text{ by } \left((e^{i\theta_1}, \cdots, e^{i\theta_d}), x\right) \mapsto (e^{i\theta_1}, \cdots, e^{i\theta_d}) \circ x.$$

Consider the fundamental vector fields in each directions of T^d by

$$T_j u(x) := \left.\frac{\partial}{\partial\theta_j}\right|_{\theta_j=0} u\left((1, \cdots, e^{i\theta_j}, \cdots, 1)\circ x\right), \text{ for all } j = 1, \cdots, d,\ x \in X.$$

We say that the group action is

(1) CR, if $[T_j, C^\infty(X, T^{1,0}X)] \subset C^\infty(X, T^{1,0}X)$ for each $j = 1, \cdots, d$.
(2) transversal, if there exists a pair $(\mu_1, \cdots, \mu_d) \in \mathbb{R}^d \setminus (0, \cdots 0)$ such that

$$T^{1,0}_x X \bigoplus T^{0,1}_x X \bigoplus \mathbb{C}(\sum_{j=1}^d \mu_j T_j)(x) = \mathbb{C}T_x X$$

for all $x \in X$.

From now on, we assume the condition for CR and transversality, and this leads to

Proposition 4.1. *$T_j\bar{\partial}_b = \bar{\partial}_b T_j$ for all $j = 1,,,d$.*

Proof. Since the transversal properties is an open condition, by the continuity we have

$$T^{1,0}X \oplus T^{0,1}X \oplus \mathbb{C}\langle \mu_{j,1}T_1 + \cdots + \mu_{j,d}T_d\rangle = \mathbb{C}TX$$

where

$$\left[\mu_{j,k}\right]_{1\le j,k\le d} = \begin{bmatrix} \mu_1 & \mu_2 & \mu_3 & \dots & \mu_d \\ \mu_1 & \mu_2+\epsilon & \mu_3+\epsilon & \dots & \mu_d+\epsilon \\ \mu_1 & \mu_2 & \mu_3+\epsilon & \dots & \mu_d+\epsilon \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mu_1 & \mu_2 & \mu_3 & \dots & \mu_d+\epsilon \end{bmatrix}$$

is an invertible matrix. Consider the vector fields

$$T_{0,j} := \sum_{k=1}^{d} \mu_{j,k}T_k,$$

which induce CR, transversal group actions. So similar to the case in Hsiao–Li [9], after selecting the canonical patch from Baouendi–Rothschild–Trèves [1], we have

$$T_{0,j}\bar{\partial}_b = \bar{\partial}_b T_{0,j} \text{ for each j} = 1,\cdots d.$$

In ohter words,

$$\left[\mu_{j,k}\right]_{1\le j,k\le d} \begin{bmatrix} T_1\bar{\partial}_b - \bar{\partial}_b T_1 \\ \vdots \\ T_d\bar{\partial}_b - \bar{\partial}_b T_d \end{bmatrix} = 0,$$

and hence T_j commutes with $\bar{\partial}_b$. □

This observation suggests a subcomplex of the $\bar{\partial}_b$-complex by taking

$$\Omega^{(0,q)}_{p_1,\cdots,p_d}(X) := \{u \in \Omega^{(0,q)}(X) : T_j u = ip_j u \text{ for all } j = 1,\cdots,d\},$$

and the restriction

$$\bar{\partial}_b : \cdots \to \Omega^{(0,q)}_{p_1,\cdots,p_d}(X) \to \Omega^{(0,q+1)}_{p_1,\cdots,p_d}(X) \to \cdots$$

We also have the torus equivariant cohomology

$$H^q_{b,p_1,\cdots,p_d}(X) := \frac{\ker \bar{\partial}_b}{\mathrm{im}\bar{\partial}_b}.$$

In this section, we aim to establish the Hodge theorem for $H^q_{b,p_1,\cdots,p_d}(X)$. The starting point is to translate our problem of torus equivariant to the case of $\mathbb{R}$ equivariant, via the following series of observations: First, as indicated in Hendrick–Hsiao–Li [8], there is

Lemma 4.1. *We may assume $(\mu_1,\cdots,\mu_d) \in \mathbb{R}^d$ are linearly independent of $\mathbb{Q}$ such that the $\mathbb{R}$-action induced by $T_0 := \sum_{j=1}^d \mu_j T_j$ is still CR and transversal.*

Proof. Suppose $\mu_1,\cdots,\mu_d$ are linear dependent over $\mathbb{Q}$, without loss of generality, we may assume $\mu_1,\cdots,\mu_k$ are linear independent over $\mathbb{Q}$, where $1 \le k < d$. Write

$$\mu_l := \sum_{j=1}^{k} r_{j,l}\mu_j,\ l = k+1,\cdots,d,\ r_{jl} \in \mathbb{Q}.$$

Consider a new torus action on X defined by

$$x \mapsto (e^{i\theta_1}, \cdots, e^{i\theta_k}) \cdot x := (e^{iN\theta_1}, \cdots, e^{iN\theta_k}, e^{iN\sum_{j=1}^{k} r_{j,k+1}\theta_j}, \cdots e^{iN\sum_{j=1}^{k} r_{j,d}\theta_j}) \circ x,$$

where $N \in \mathbb{N}$ is the least integer such that $r_{j,l} \big| N$ for all $j = 1, \cdots k$, $l = k+1 \cdots d$. Take the vector fields $\mathscr{T}_j$ on $\mathscr{C}^\infty(X)$ by

$$\mathscr{T}_j := \frac{\partial}{\partial \theta_j}\bigg|_{\theta_j = 0} u\left((1, \cdots, e^{i\theta_j}, \cdots, 1) \cdot x\right), \text{ for all } u \in \mathscr{C}^\infty(X)$$

for all $j = 1, \cdots, k$, then it is clear that

$$T_0 := \sum_{j=1}^{d} \mu_j T_j = \sum_{j=1}^{k} \frac{\mu_j}{N} \mathscr{T}_j,$$

where $\frac{\mu_1}{N}, \cdots, \frac{\mu_k}{N}$ are real numbers linear independent over $\mathbb{Q}$. Also, by construction of T_0, we have

$$\left[T_0, \mathscr{C}^\infty(X, T^{1,0}X)\right] \subset \mathscr{C}^\infty(X, T^{1,0}X),$$

and

$$T_0(x) \oplus T_x^{1,0}X \oplus T_x^{0,1}X = \mathbb{C}T_xX \text{ for all } x \in X,$$

i.e. the induced $\mathbb{R}$-action is also CR and transversal. □

Second, we wish to understand the spectral of $-iT_0$. Consider a T_0-rigid L^2 inner product $(\cdot|\cdot)$ induced by the CR, transversal of the $\mathbb{R}$-action, then we have:

Proposition 4.2. *The operator $-iT_0 : \Omega^{(0,q)}(X) \to \Omega^{(0,q)}(X)$ has a self adjoint bounded extension*

$$-iT_0 : \text{Dom}\,(-iT_0) \subset L^2_{(0,q)}(X) \to L^2_{(0,q)}(X)$$

where the domain is defined by Dom $(-iT_0) := \{u \in L^2_{(0,q)}(X) : -iT_0 u \in L^2_{(0,q)}(X)\}$.

Proof. As in the case for Hsiao–Li [9], since T_0 is transversal, locally we have $T_0 = \frac{\partial}{\partial \eta}$ locally; moreover, if we fix the T_0-rigid inner product $(\cdot|\cdot)$, which induces the volume form with local expression

$$dV_X(x) = 2^n \lambda(x_1, \cdots, x_{2n}) dx_1 \cdots dx_{2n} d\eta \text{ for a real-valued smooth function } \lambda$$

then for $u, v \in \Omega^{(0,q)}(X)$, on the canonical patch $D \subset X$, we have

$$\begin{aligned}
(-iT_0 u|v) &= \int_D (-i\frac{\partial}{\partial \eta} u)\bar{v} 2^n \lambda(x_1, \cdots, x_{2n}) dx_1 \cdots dx_{2n} d\eta \\
&= \int_D u\left(-i\frac{\partial}{\partial \eta} \bar{v} 2^n \lambda(x_1, \cdots, x_{2n})\right) dx_1 \cdots dx_{2n} d\eta \\
&= \int_D u \overline{-i\frac{\partial}{\partial \eta} v} 2^n \lambda(x_1, \cdots, x_{2n}) dx_1 \cdots dx_{2n} d\eta \\
&= (u| - iT_0 v).
\end{aligned}$$

Apply the Friedrich's lemma for the first order differential operator $-iT_0$, we can extend the above equation to the case of $u, v \in L^2_{(0,q)}(X)$, by considering $\{u_j\}, \{v_j\} \in \Omega^{(0,q)}(X)$ such that

$$u_j \to u,\ v_j \to v\ , -iT_0 u_j \to -iT_0 u,\ -iT_0 v_j \to -iT_0 v \text{ in } L^2_{(0,q)}(X).$$

Then

$$(-iT_0u|v) = \lim_{j\to\infty}(-iT_0u_j|v_j) = \lim_{j\to\infty}(u_j|-iT_0v_j) = (u|-iT_0v).$$

So we get the symmetry condition for $-iT_0$.

Next, on one hand, recall that

$$\text{Dom }(-iT_0)^* := \{v \in L^2_{(0,q)}(X) : \exists\, c > 0 \text{ s.t. } \left|(-iT_0u|v)\right| < c\|u\| \text{ for all } u \in \text{Dom }-iT_0\}$$

where $\|u\|^2_{L^2} := (u|u)$.

So by Riesz's lemma, for all $v \in \text{Dom }(-iT_0)^*$, there exists $w \in L^2_{(0,q)}(X)$ such that

$$(-iT_0u|v) = (u|w) \text{ for all } u \in \text{Dom }(-iT_0)$$

Since this also holds for all $u \in \Omega^{(0,q)}(X)$, along with the observation that $-iT_0$ is symmetric, it implies $-iT_0v = w \in L^2_{(0,q)}(X)$, i.e. Dom $(-iT_0) \subset$ Dom $(-iT_0)^*$.

On the other hand, by Cauchy-Schwartz inequality and the symmetry again, we can also find Dom $(-iT_0) \subset$ Dom $(-iT_0)^*$.

In conclusion, $-iT_0 = (-iT_0)^*$. $\square$

Proposition 4.3. *Fix a $(p_1, \cdots, p_d) \in \mathbb{Z}^d$, let $p_\beta := \sum_{j=1}^d \mu_j p_j$, where $\{\mu_j\}_{j=1}^d$ is chosen in* **Lemma 4.1**, *then the L^2 eigenspace of $-iT_0$ is*

$$L^2_{(0,q),p_\beta}(X) := \{u \in \text{Dom }(-iT_0) : -iT_0u = p_\beta u\} = L^2_{(0,q),p_1,\cdots,p_d}(X) \neq \{0\}.$$

Proof. First, we show that for any fixed $(p_1, \cdots, p_d) \in \mathbb{Z}^d$, $L^2_{(0,q),p_1,\cdots,p_d}(X) \neq \{0\}$. The idea is to use rational approximation to reduce to the case for circle action. Choose $(\gamma_1, \cdots, \gamma_d) \in \mathbb{Q}^d$ closed enough to $(\mu_1, \cdots, \mu_d) \in \mathbb{R}^d$, and consider the vector field $\hat{T}_0 := \sum_{j=1}^d \gamma_j T_j$, which induces a CR, transversal S^1-action, and after some proper scaling, we may assume it has period of 2π. Denote such S^1-action of period 2π by $S^1 \times X \ni (e^{i\theta}, x) \mapsto e^{i\theta} \cdot x \in X$ (we use the notation $\cdot$ to distinguish the one for group action $\circ$ appeared earlier), let

$$X_{\text{reg}} := \{x \in X : e^{i\theta} \cdot x \neq x \text{ for all } \theta \in [0, 2\pi)\}.$$

Note that the regular set is non-empty. For $p_\gamma := \sum_{j=1}^d \gamma_j p_j$, there is

$$L^2_{(0,q),p_1,\cdots,p_d}(X) \supset \Omega^{(0,q)}_{p_1,\cdots,p_d} X = \hat{\Omega}^{(0,q)}_{p_\gamma} X := \{u \in \Omega^{(0,q)}(X) : \hat{T}_0 u = ip_\gamma u\}.$$

So for $x := (x', x_{2n+1}) \in X_{\text{reg}}$, apply the **Theorem 3.1**, locally on the canonical patch D, we have

$$e^{i\theta} \cdot (x', x_{2n+1}) \notin D \text{ for all } \theta \in (\epsilon, 2\pi - \epsilon)$$

and

$$T_0 = \frac{\partial}{\partial x_{2n+1}} \text{ on } D.$$

Hence, consider $\chi(x) \in \mathscr{C}_0^\infty(D)$ such that $\int_X \chi(x', x_{2n+1}) dx_{2n+1} \neq 0$, and for $|J| = q$, take

$$u_J(x) := \chi(x) e^{ip_\gamma x_{2n+1}} \in \mathscr{C}_0^\infty X,$$

then

$$\hat{\Omega}^{(0,q)}_{p_\gamma} X \ni \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta} \cdot x) e^{-ip_\gamma\theta} d\theta = \int_X \chi_x dx_{2n+1} \neq 0,$$

so this part is done.

Second, for a fixed $p \in \mathbb{Z}^d$, by the definition of p_β, the direct computation gives $L^2_{(0,q),p_1,\cdots,p_d}(X) \subset E_{p_\beta}$. Conversely, suppose $L^2_{(0,q),p_1,\cdots,p_d}(X) \subsetneq E_{p_\beta}$, then there exists a $u \in E_{p_\beta}$, $\|u\| = 1$, $u \perp L^2_{(0,q),p_1,\cdots,p_d}(X)$.

Note that $-iT_0(Q^q_{m_1,\cdots,m_d}u) = (\sum_{j-1}^d \mu_j m_j)Q^q_{m_1,\cdots,m_d}u$ for all $m \in \mathbb{Z}^d$, and by the assumption that $\{\mu_j\}_{j=1}^d$ are linear independent over $\mathbb{Q}$, we have

$$p_\beta := \sum_{j=1}^d \mu_j p_j = \sum_{j=1}^d \mu_j m_j \Leftrightarrow m = p.$$

This implies

$$(u|Q^{(q)}_{m_1,\cdots,m_d}u) = \begin{cases} 0: & \text{if } m = p \text{ by the perpendicular assumption} \\ 0: & \text{if } m \neq p \text{ because the intersection of two eigenspaces is null} \end{cases}.$$

In conclusion, for $N \in \mathbb{N}$,

$$\|u - \sum_{m\in\mathbb{Z}^d,|m|\leq N} Q^{(q)}_{m_1,\cdots,m_d}u\|^2_{L^2} = \|u\|^2_{L^2} + \sum_{m\in\mathbb{Z}^d,|m|\leq N} \|Q^{(q)}_{m_1,\cdots,m_d}u\|^2_{L^2} \geq 1.$$

However, the L.H.S tends to zero by the theory of Fourier series, this makes a contradiction. □

Theorem 4.1.

(1) $\lambda \in \text{Spec}\,(-iT_0) \Leftrightarrow \lambda$ *is an* L^2 *eigenvalue of the form* $\lambda = \sum_{j=1}^d \mu_j p_j$ *in* **Proposition** 4.3.

(2) $H^q_{b,p_1,\cdots,P_d}(X) \cong \ker \Box^{(q)}_{b,p_1,\cdots,p_d} := \{u \in \text{Dom}\,\Box^{(q)}_{b,p_1,\cdots,p_d} : \Box^{(q)}_b u = 0\}$ *is a finite dimensional subspace of* $\Omega^{(0,q)}_{p_1,\cdots,p_d}(X)$.

Proof.

(1) We here use some general spectral theory of self-adjoint operators (cf. Davies, E. B. [5, Chapter 2]). Since $-iT_0 : \text{Dom}\,(-iT_0) \subset L^2_{(0,q)}(X) \to L^2_{(0,q)}(X)$ is a self adjoint operator, then $S := \text{Spec}\,F \subset \mathbb{R}$ and there exists a positive finite measure $d\mu$ on $S \times \mathbb{N}$ such that

$$L^2_{(0,q)}(X) \cong \left\{h(s,n) : S \times \mathbb{N} \to \mathbb{R} \middle| \int |h|^2 d\mu \leq \infty\right\}.$$

Under this isomorphism, we realize $-iT_0$ as

$$-iT_0 : h(s,n) \in \text{Dom}\,(-iT_0) \mapsto sh(s,n) \in L^2(S \times \mathbb{N}, d\mu)$$

where

$$\text{Dom}\,(-iT_0) \cong \left\{h(s,n) \in L^2(S \times \mathbb{N}, d\mu) \middle| \int |sh(s,n)|^2 < \infty\right\}.$$

Now consider $A := \{\sum_{j=1}^d \mu_j p_j : (p_1,\cdots,p_d) \in \mathbb{Z}^d\}$ and a Borel set $B \subset \mathbb{R}$ such that $B \cap A = \emptyset$. Take the spectral projection of B, which can be seen as

$$E(B) : h(s,n) \mapsto 1_B(s)h(s,n).$$

Then note that for $g \in \text{Rang}\,E(B)$, by $Q^q_{(m_1,\cdots,m_d)}g \subset \text{Rang}\,E(A) \cap \text{Rang}\,E(B) = \{0\}$, there is

$$\forall (m_1,\cdots,m_d) \in \mathbb{Z}^d,\ (g|Q^q_{m_1,\cdots,m_d}g) = 0.$$

Finally, along with the approximation by Fourier series, we have

$$\|g\|_{L^2}^2 + \|Q^q_{m_1,\cdots,m_d} g\|_{L^2}^2 = \|g - Q^q_{m_1,\cdots,m_d} g\|_{L^2}^2 \to 0.$$

So $g = 0$.

(2) Since $\Box_b^{(q)} - T_0^2$ is a self adjoint elliptic operators, we can repeat the proof of **Theorem 3.2**.

$\square$

4.2. **The method of rational approximation.** With the study of the operator $-iT_0$ previously, we are now ready to approximate the $\mathbb{R}$-action induced by T_0 by some sequence of S^1-action, and get the asymptotic bounds for torus equivariant CR sections.

For a fixed $(p_1, \cdots, p_d) \in \mathbb{Z}^d$, consider the corresponding $\alpha := \sum_{j=1}^d \mu_j p_j \in \text{Spec}\,(-iT_0)$. Then for each $j = 1,,,.d$, consider a sequence $\{\mu_{k,j}\}_{k=1}^\infty$ such that $\mu_{k,j} \in \mathbb{Q}$ converges to μ_j, let $\hat{T}_k := \sum_{j=1}^d \mu_{k,j} T_j \to T_0$ and $\alpha_k := \sum_{j=1}^d \mu_{k,j} p_j \to \alpha$ as $k \to \infty$. For all $m \in \mathbb{N}$, put the space of tours equivaraint CR sections

$$\mathcal{H}^0_{b,m\alpha}(X) := \{u \in \ker \Box_b^{(0)} : -iT_0 u = m\alpha u\}$$

and

$$\mathscr{H}^q_{b,m\alpha_k}(X) := \{u \in \ker \Box_b^{(0)} : -i\hat{T}_k u = m\alpha_k u\}.$$

Note that for all $m \in \mathbb{N}$, we already have

$$\mathcal{H}^q_{b,m\alpha}(X) \subset \mathscr{H}^q_{b,m\alpha_k}(X).$$

However, the other inclusion may not happen , because the issue

$$\sum_{j=1}^d \mu_{k,j} \hat{p}_j = \sum_{j=1}^d \mu_{k,j} p_j \text{ for another } \hat{p} \neq p \tag{4.1}$$

occurs. Accordingly, it's crucial to specify what kinds of lattice point $(p_1, \cdots, p_d)$ makes

$$\dim_{\mathbb{C}} \mathcal{H}^q_{b,m\alpha}(X) = \dim_{\mathbb{C}} \mathscr{H}^q_{b,m\alpha_k}(X),$$

which answer the question when does $\dim_{\mathbb{C}} \mathcal{H}^q_{b,m\alpha}(X) = O(m^n)$.

We now illustrate how we locate such lattice point: For a fixed $(p_1, \cdots, p_d) \in \mathbb{Z}^d$, suppose $\mu_j p_j > 0$ for all $j = 1, \cdots, d$, and consider the case

$$H^0_{b,p_1,\cdots,p_d}(X) = \ker \Box^{(0)}_{b,\alpha} := \{u \in \ker \Box_b^{(0)} : -iT_0 u = \alpha u\} \neq \{0\},$$

where $\alpha := \sum_{j=1}^d \mu_j p_j$. Then $\forall m \in \mathbb{N}$, $m \geq 2$, we also have $m\mu_j p_j > 0$ for all j, and

$$\ker \Box^{(0)}_{b,m\alpha} := \{u \in \ker \Box_b^{(0)} : -iT_0 u = m\alpha u\} \neq \{0\}.$$

Hence, $\forall m \in \mathbb{N}$, one one hand we have $m\alpha > 0$. On the other hand, $m^2\alpha^2 \in \text{Spec}\,(\Box_b^{(q)} - T_0^2)$, which is a discrete subset of $\mathbb{R}^+$ by the ellipticity and positivity of $\Box_b^{(q)} - T_0^2$. So for each $m \in \mathbb{N}$, there exists a constant $C_m > 0$ such that

$$\inf\{|m^2\alpha^2 - \beta| : \beta \in \text{Spec}\,(\Box_b^{(0)} - T_0^2)\} = C_m > 0.$$

However, this is not enough to guarantee the method of rational approximation works. In fact, under a certain spectral gap assumption, we have the following key lemma:

Lemma 4.2. *For a fixed $(p_1, \cdots, p_d) \in \mathbb{Z}^d$ of $H^0_{b,p_1,\cdots,p_d}(X) \neq \{0\}$, assume*

$$\mu_j p_j > 0 \text{ for all } j = 1, .., d$$

and there exists a constant $c > 0$ such that

$$\inf\left\{|m^2\alpha^2 - \beta^2| : \beta \in \text{Spec}\,(-iT_0),\ \beta \neq m\alpha,\ \ker \Box^{(0)}_{b,\beta} \neq \{0\},\ m \in \mathbb{N}\right\} = C > 0.$$

Then for all $m \in \mathbb{N}$, there exists a large $k_0 \in \mathbb{N}$ independent of m such that for all $k \geq k_0$, the orthogonal projection

$$u \in \{u \in \ker \Box^{(0)}_b : \hat{T}_k u = im\alpha_k u\} \mapsto Q^{(0)}_{m\alpha} u \in \{u \in \ker \Box^{(0)}_b : T_0 u = im\alpha u\}$$

with respect to the T_0 invariant L^2 inner product $(\cdot|\cdot)$ is bijective.

Proof. The surjectivity holds because it is a projection map. For the injectivity, suppose otherwise, then for each $k \in \mathbb{N}$,

$$\exists\, u_k \in \ker \Box^{(0)}_b \text{ such that } \hat{T}_k u_k = im\alpha_k u \text{ and } \|u_k\|^2_{L^2} = 1 \text{ with } Q^{(0)}_{m\alpha} u_k = 0.$$

Since T_0 is self adjoint, $u_k \in \ker \Box^{(0)}_b \subset \Omega^{(0,q)}(X)$ and $Q^{(0)}_\alpha u_k = 0$, we can take the orthogonal decomposition

$$u_k = \sum_{l=1}^{\infty} u_{k,l} \text{ in } C^\infty \text{ topology, with } \|u_k\|^2_{L^2} = \sum_{l=1}^{\infty} \|u_{k,l}\|^2_{L^2}.$$

Here, $T_0 u_{k,l} = i\beta_{k,l} u_{k,l}$, $\beta_{k,l} \neq m\alpha$ for all l. Note that by $\Box^{(0)}_b T_0 = T_0 \Box^{(0)}_b$, we have $u_{k,l} \in \ker \Box^{(0)}_b$, so $\beta^2_{k,l} \in \text{Spec}\,(\Box^0_b - T_0^2)$. For simplicity, we let $\beta_{k,1} := -m\alpha$. Now, we use the estimate on Sobolev norm of u_k to reach a contradiction. First, since $\hat{T}_k \to T_0$, there exists $\epsilon_k \to 0$ such that

$$\begin{aligned}
\epsilon_k \|u_k\|^2_{H^1} &= \|(T_0 - \hat{T}_k) u_k\|^2_{L^2} \\
&= \sum_{l=1}^{\infty} |\beta_{k,l} - m\alpha_k|^2 \|u_{k,l}\|^2_{L^2} \\
&\geq |\beta_{k,1} - m\alpha_k|^2 \|u_{k,1}\|^2_{L^2} \\
&> m^2\alpha^2 \|u_{k,1}\|^2_{L^2} \text{ when } k \gg 1 \text{ makes } \alpha_k > 0.
\end{aligned}$$

By the a Gårding's inequality for the second order strongly elliptic operator $\Box^{(q)}_b - T_0^2$, along with the use of $\hat{T}_k \to T_0$ again, there is a constant C_1 such that

$$\begin{aligned}
\|u_k\|^2_{H^1} &\leq C_1 \left(\text{Re}\left((\Box^{(q)}_b - T_0^2) u_k \middle| u_k\right) + \|u_k\|^2_{L^2}\right) \\
&= C_1 \left(\|T_0 u_k\|^2_{L^2} + \|u_k\|^2_{L^2}\right) \\
&= C_1 \left(\|(T_0 - \hat{T}_k + \hat{T}_k) u_k\|^2_{L^2} + \|u_k\|^2_{L^2}\right) \\
&\leq C_1 \left(\epsilon_k \|u_k\|^2_{H^2} + \|\hat{T}_k u_k\|^2_{L^2} + \|u_k\|^2_{L^2}\right) \\
&\leq C_1 \epsilon_k \|u_k\|^2_{H^1} + C_1(m^2\alpha_k^2 + 1).
\end{aligned}$$

Second, since we also have $\hat{T}_k^2 \to T_0^2$, there exists $\tilde{\epsilon}_k \to 0$ such that

$$\begin{aligned}\tilde{\epsilon}_k \|u_k\|_{H^2}^2 &= \|(T_0^2 - \hat{T}_k^2)u_k\|_{L^2}^2 \\ &= \|\sum_{l=1}^{\infty}(m^2\alpha_k^2 - \beta_{k,l}^2)u_{k,l}\|_{L^2}^2 \\ &\geq \sum_{l=2}^{\infty}\left|(m^2\alpha_k^2 - m^2\alpha^2) - (\beta_{k,l}^2 - m^2\alpha^2)\right|^2 \|u_{k,l}\|_{L^2}^2 \\ &\geq \sum_{l=2}^{\infty}\left||\beta_{k,l}^2 - m^2\alpha^2| - |m^2\alpha_k^2 - m^2\alpha^2|\right|^2 \|u_k^l\|_{L^2}^2 \\ &> (2m^2-1)^2C^2\sum_{l\geq 2}\|u_k\|_{L^2}^2 \text{ when } k \gg 1 \text{ makes } |\alpha_k^2 - \alpha^2| < 2C.\end{aligned}$$

By the a priori estimate for the second order elliptic operator $\Box_b^{(q)} - T_0^2$, along with the use of $\hat{T}_k^2 \to T_0^2$ again, there is a constant C_2 such that

$$\begin{aligned}\|u_k\|_{H^2}^2 &\leq C_2\left(\left\|(\Box_b^{(q)} - T_0^2)u_k\right\|_{L^2}^2 + \|u_k\|_{L^2}^2\right) \\ &= C_2\left(\|T_0^2u_k\|_{L^2}^2 + \|u_k\|_{L^2}^2\right) \\ &= C_2\left(\|(T_0^2 - \hat{T}_k^2 + \hat{T}_k^2)u_k\|_{L^2}^2 + \|u_k\|_{L^2}^2\right) \\ &\leq C_2\left(\tilde{\epsilon}_k\|u_k\|_{H^2}^2 + \|\hat{T}_k^2u_k\|_{L^2}^2 + \|u_k\|_{L^2}^2\right) \\ &\leq C_2\tilde{\epsilon}_k\|u_k\|_{H^2}^2 + C_2(m^4\alpha_k^4 + 1)\end{aligned}$$

Hence, if we denote $k_0 \gg 1$ to be the number such that all the above estimate holds, then

$$\begin{aligned}1 &= \|u_{k_0}\|_{L^2}^2 \\ &= \|u_{k_0,1}\|_{L^2}^2 + \sum_{l\geq 2}\|u_{k_0,l}\|_{L^2}^2 \\ &\leq \frac{\epsilon_{k_0}}{m^2\alpha^2}\|u_{k_0}\|_{H^1}^2 + \frac{\tilde{\epsilon}_{k_0}}{m^4C^2}\|u_{k_0}\|_{H^2}^2 \\ &\leq \frac{\epsilon_{k_0}}{m^2\alpha^2}\frac{C_2(m^2\alpha_{k_0}^2+1)}{1-C_1\epsilon_{k_0}} + \frac{\tilde{\epsilon}_{k_0}}{(2m^2-1)^2C^2}\frac{C_2(m^4\alpha_{k_0}^4+1)}{1-C_2\tilde{\epsilon}_{k_0}} \\ &\leq \frac{\epsilon_{k_0}}{\alpha^2}\frac{C_2(\alpha_{k_0}^2+1)}{1-C_1\epsilon_{k_0}} + \frac{\tilde{\epsilon}_{k_0}}{C^2}\frac{C_2(\alpha_{k_0}^4+1)}{1-C_2\tilde{\epsilon}_{k_0}} \ll 1 \text{ if we take } k_0 \text{ larger.}\end{aligned}$$

and this leads to a contradiction. □

Under the same condition of the lemma, since $\mathscr{H}_{b,m\alpha_k}^q(X)$ has finite dimension for all $m \in \mathbb{N}$, by rank-nullity theorem we can conclude that:

Theorem 4.2. *For a fixed $(p_1, \cdots, p_d) \in \mathbb{Z}^d$ of $H_{b,p_1,\cdots,p_d}^0(X) \neq \{0\}$, assume*

$$\mu_j p_j > 0 \textit{ for all } j = 1, .., d$$

and suppose that there exists a constant $c > 0$ such that

$$\inf\left\{|m^2\alpha^2 - \beta^2| : \beta \in \operatorname{Spec}(-iT_0),\ \beta \neq m\alpha,\ \ker\Box_{b,\beta}^{(0)} \neq \{0\},\ m \in \mathbb{N}\right\} = C > 0.$$

Then for all $m \in \mathbb{N}$, there exists a large $k_0 \in \mathbb{N}$ independent of m such that for all $k \geq k_0$

$$\dim_{\mathbb{C}} \mathcal{H}^0_{b,m\alpha}(X) = \dim_{\mathbb{C}} \mathscr{H}^0_{b,m\alpha_k}(X).$$

4.3. **The torus equivariant CR Siu–Demailly–Grauert–Riemenschneider criterion.** Fix a lattice point $(p_1, \cdots, p_d) \in \mathbb{Z}^d$ and the k_0 as in **Theorem 4.2**. Then for all $m \in \mathbb{N}$, first we apply **Theorem 2.3** to the S^1-equivaraint tangential Cauchy–Riemann complex $\left(\Omega^{(0,\bullet)}_{m\alpha_{k_0}}(X), \hat{\partial}_{b,m\alpha_{k_0}}\right)$ induced by $\hat{T}_{k_0}$ (note that the period of this circle action may not be 2π), that is

$$\sum_{j=0}^{n}(-1)^j \dim_{\mathbb{C}} \hat{\mathscr{H}}^j_{b,m\alpha_{k_0}}(X) = \frac{1}{2\pi}\int_X \mathrm{Td}_b(T^{1,0}X) \wedge \exp(-\frac{m\alpha_{k_0} d\hat{\omega}_0}{2\pi}) \wedge \hat{\omega}_0. \tag{4.2}$$

(Here $\hat{\omega}_0$ is the canonical one form dual to $\hat{T}_{k_0}$; $\mathrm{Td}_b(T^{1,0})X$ is the T_0-rigid, and hence the T_{k_0}-rigid tangential Todd class on X; $\hat{\mathscr{H}}^j_{b,m\alpha_{k_0}}(X) := \{u \in \ker \hat{\Box}^{(j)}_b : -i\hat{T}_{k_0} u = m\alpha_{k_0} u\}$, where $\hat{\Box}^{(j)}_b$ is the Kohn Laplacian determined by $\hat{T}_{k_0}$). Second, by **Theorem 2.2**, we also have

$$\dim_{\mathbb{C}} \hat{\mathscr{H}}^j_{b,m\alpha_{k_0}}(X) \leq \frac{(m\alpha_{k_0})^n}{2\pi^{n+1}} \int_{\hat{X}(j)} |\det \hat{\mathcal{L}}_x| dV_X(x) + o(m^n). \tag{4.3}$$

(Here, $\hat{\mathcal{L}}_x$ is the Levi form induced by T_{k_0}, and recall that the index set $\hat{X}(q) := \{x \in X : \hat{\mathcal{L}}_x$ is nondegenerate and has exactly q negative eigenvalues$\}$). Third, since the CR structure $T^{1,0}X$ is fixed, we have

$$\ker \Box^{(0)}_b = \ker \bar{\partial}_b = \ker \hat{\bar{\partial}}_b = \ker \hat{\Box}^{(0)}_b$$

so

$$\mathcal{H}^0_{b,m\alpha}(X) = \mathscr{H}^0_{b,m\alpha_{k_0}}(X) = \hat{\mathscr{H}}^0_{b,m\alpha_{k_0}}(X). \tag{4.4}$$

In conclusion, when X is a torus invariantly weakly pseudoconvex and torus invariantly pseudoconvex at a point; in other words, the Siu's type condition that $\mathcal{L}_x \geq 0$ for all $x \in X$ and $\mathcal{L}_p > 0$ for some $p \in X$ holds. Then,

$$\dim_{\mathbb{C}} \mathcal{H}^0_{b,m\alpha}(X) = \dim_{\mathbb{C}} \mathscr{H}^0_{b,m\alpha_{k_0}}(X) = \dim_{\mathbb{C}} \hat{\mathscr{H}}^0_{b,m\alpha_{k_0}}(X) = O(m^n). \tag{4.5}$$

We reason (4.5) as follows: By construction,

$$T_0 = \gamma(x)\hat{T}_{k_0} \bmod T^{1,0}X \oplus T^{0,1}X$$

for some $\gamma(x) > 0$, and

$$\omega_0(x) = \frac{1}{\gamma(x)}\hat{\omega}_0(x) \bmod T^{*1,0}X \oplus T^{*0,1}X.$$

The Cartan's formula gives

$$\mathcal{L}_{ij}(x) = \langle \omega_0(x), [Z_j, \overline{Z}_k] \rangle = \left\langle \frac{1}{\gamma(x)}\hat{\omega}_0(x), [Z_j, \overline{Z}_k] \right\rangle = \frac{1}{\gamma(x)}\hat{\mathcal{L}}_{ij}(x)$$

where $\{Z_j\}_{j=1}^n$ is a basis of $T^{1,0}_x X$. This leads to $\hat{\mathcal{L}}_x \geq 0$ for all $x \in X$ and $\hat{\mathcal{L}}_p > 0$ for some $p \in X$, hence on one hand

$$\hat{X}(0) \text{ is containted in a ball, } \hat{X}(q) = \emptyset \text{ for all } q \geq 1$$

or equivalently

$$\dim_{\mathbb{C}} \hat{\mathscr{H}}^j_{b,m\alpha_{k_0}}(X) = o(m^n) \text{ for all } j \geq 1 \text{ by Theorem 2.2.}$$

On the other hand, since $\alpha > 0$ by our assumption, we also have

$$\sum_{j=1}^{n}(-1)^j \dim_{\mathbb{C}} \mathscr{H}^j_{b,m\alpha_{k_0}}(X) = O(m^n) \text{ by Theorem 2.3.}$$

We summarize the above discussion as our main theorem:

Theorem 4.3 (The torus equivariant CR Siu–Demailly–Grauert–Riemenschneider criterion)**.** *Let X be a CR manifold endowed with a transversal, CR torus action on X. Assume X is torus invariantly pseudoconvex and torus invariantly strongly pseudoconvex at a point. For a fixed $(p_1, \cdots, p_d) \in \mathbb{Z}^d$ such that $H^0_{b,p_1,\cdots,p_d}(X) \neq \{0\}$, assume*

$$\mu_j p_j > 0 \textit{ for all } j = 1, \cdots, d$$

and suppose that there exists a constant $C > 0$ such that

$$\inf\left\{|m^2\alpha^2 - \beta^2| : \beta \in \text{Spec}\,(-iT_0),\ \beta \neq m\alpha,\ \ker \Box^{(0)}_{b,\beta} \neq \{0\},\ m \in \mathbb{N}\right\} = C > 0.$$

(where $\alpha := \sum_{j=1}^d \mu_j p_j$, $\{\lambda_j\}_{j=1}^d$ are the transversal data linearly independent over $\mathbb{Q}$) then

$$\dim_{\mathbb{C}} H^0_{b,mp_1,\cdots,mp_d}(X) = \dim_{\mathbb{C}} H^0_{m\alpha}(X) = O(m^n)$$

We end this section by the torus equivariant Siu–Demailly–Grauert–Riemenschneider criterion on complex manifolds. Let M be a compact complex manifold of $\dim_{\mathbb{C}} M = n$ with a holomorphic torus action, and (L, h^L) be a holomorphic line bundle over M with a torus invariant smooth hermitian metric. Since the torus action is holomorphic and L is also h, we have $\bar{\partial} T_j = T_j \bar{\partial}$ for all $j = 1, \cdots d$, where T_j are the fundamental vector fields in j-th direction induced by the torus action. So we can take the space of torus equivariant holomorphic section for $(p_1, \cdots, p_d) \in \mathbb{Z}^d$ by

$$H^0_{p_1,\cdots,p_d}(M, L) := \{u \in \mathscr{C}^\infty(M, L) : \bar{\partial} u = 0,\ -iT_j u = p_j u\}.$$

Consider the circle bundle $X := \{v \in L^* : |v|^2_{h^{L^*}} = 1\}$, which is a CR manifold endowed with a natural CR, transversal S^1-action on its fiber. We can check that:

(1) The induced tours action

$$T^d \times S^1 = T^{d+1} \curvearrowright X$$

also satisfies the CR, transversal properties. In fact, for a fixed $(p_1, \cdots, p_d) \in \mathbb{Z}^d$, the transversal data $(\mu_1, \cdots, \mu_{d+1}) \in \mathbb{R}^{d+1}$ can be choose to be any pair of real numbers of the form $(\mu_1, \cdots, \mu_d, 1)$. As for the CR condition, since the projection $\pi^X : X \to M$ is a submersion, for all j we can lift T_j to X, and denote them as $\tilde{T}_j$. Combine the π^X-relatedness of T_j and $\tilde{T}_j$, $[\bar{\partial}_b, T_j] = 0$ and the assumption that h^L is tours invariant, we get $\tilde{T}_j \bar{\partial}_b = \bar{\partial} \tilde{T}_j$. (Here, one way to understand the assumption that h^L is tours invariant is by taking the local picture. In the canonical patch of Theorem 3.1 with respect to T_0, we say h^L is tours invariant if $T_j \phi = 0$ for all $j = 1, \cdots d$. In this patch we can also write $\bar{\partial}_b = \sum_{j=1}^n \left(\frac{\partial}{\partial \bar{z}_j} - i\frac{\partial \phi(z)}{\partial \bar{z}_j}\frac{\partial}{\partial \theta}\right)$. Because T_j are induced by a holomorphic action and $\tilde{T}_j$ is π^X-related to T_j, we can find $[\tilde{T}_j, \frac{\partial}{\partial \bar{z}_j}] = 0$, i.e. $[\bar{\partial}_b, \tilde{T}_j] = 0$ for all j).

(2) X preserves the positivity of L by $R^L_z = \frac{1}{2}\mathcal{L}_x$, where $x := (z, \eta)$. In fact, let U be any local trivialization of L, and $s : U \to L$ be the local trivializing section. Take $\phi : U \to \mathbb{R}$ such that $|s(z)|^2_{h^L} = e^{-2\phi(z)}$ for some real-valued smooth function ϕ, then the canonical curvature is a real positive $(1,1)$ form locally defined by $R^L := 2\partial\bar{\partial}\phi = 2\sum_{j,k=1}^n \frac{\partial^2\phi}{\partial z_j \partial \bar{z}_k} dz_j \wedge d\bar{z}_k$. (Note that for another local trivializing section $\tilde{s} = gs$, where $g \neq 0$ is holomorphic, then $|\tilde{s}|^2_{h^L} = e^{-2\tilde{\phi}} = |g|^2|s|^2_{h^L} = e^{-2\phi+2\log|g|}$ namely $\tilde{\phi} = \phi - \log|g|$, where $\log|g|$ is harmonic. So $\partial\bar{\partial}\phi = \partial\bar{\partial}\tilde{\phi}$). Now, for (z, η) in a canonical patch of X, there is a real-valued smooth function $\phi(z)$ as in Theorem (3.1) such that

$$\begin{aligned}
\mathcal{L}_x &:= \frac{-1}{2i}\omega_0(z,\eta) \\
&= \frac{-1}{2i} d\left(d\eta - i\sum_{j=1}^n \left(\frac{\partial \phi}{\partial z_j} dz_j - \frac{\partial\phi}{\partial \bar{z}_j} d\bar{z}_j\right)\right) \\
&= \sum_{j,k=1}^n \frac{\partial^2 \phi}{\partial z_j \partial \bar{z}_k} dz_j \wedge d\bar{z}_k \\
&= \frac{1}{2} R^L_z.
\end{aligned}$$

(3) $\forall m \in \mathbb{N}$, $H^0_{p_1,\cdots,p_d}(M, L^m) \cong H^0_{p_1,\cdots,p_d,m}(X)$. This can be check as in Cheng–Hsiao–Tsai [4]: with the same local picture in the last part, let

$$A^{(q)}_{mp_1,\cdots,mp_d} : \Omega^{0,q}_{mp_1,\cdots mp_d,m}(X) \to \Omega^{0,q}_{mp_1,\cdots mp_d,m}(M, L^m)$$

by

$$u(z,\alpha) = e^{im\theta} \to s^m(z) e^{m\phi(z)} \tilde{u}(z),$$

where $\tilde{u}(z) \in \Omega^{0,q}_{mp_1,\cdots,mp_d}(U)$. After some straight forward computation, we can find that $A^{(q)}_{mp_1,\cdots,mp_d,m}$ is well-defined and bijective. Also, we can verify

$$A^{(q)}_{mp_1,\cdots mp_d,m}\bar{\partial} = \bar{\partial}_b A^{(q+1)}_{mp_1,\cdots mp_d,m}.$$

So the isomorphism follows.

Combine all the above facts, we can hence conclude a Siu's type criterion for the bigness of line bundle L in the sense of torus equivariance by using Theorem 4.3:

Corollary 4.1 (The torus equivariant Siu–Demailly–Grauert–Riemenschneider criterion)**.** *Let M be a compact complex manifold of $\dim_{\mathbb{C}} M = n$ with a holomorphic torus action T^d, and (L, h^L) be a holomorphic line bundle over M with a torus invariant smooth hermitian metric. Take any real numbers $\{\mu_j\}_{j=1}^d$ linearly independent over $\mathbb{Q}$. If the canonical curvature R^L induced by h^L satisfies $R^L \geq 0$ and $R^L_z > 0$ for some $z \in M$, and suppose that for the given lattice point $(p_1, \cdots, p_d) \in \mathbb{Z}^d$ satisfies*

$$\mu_j p_j > 0 \text{ for all } j = 1, \cdots, d,$$

and a spectral gap such that for all $m \in \mathbb{N}$, and all $(\hat{p}_1, \cdots, \hat{p}_{d+1}) \neq (mp_1, \cdots, mp_d, m)$ with

$$\ker \square^{(0)}_{\hat{p}_{d+1}}\Big|_{\mathscr{C}^\infty_{\hat{p}_1,\cdots,\hat{p}_d}(M, L^{\hat{p}_{d+1}})} \neq \{0\},$$

there is

$$\inf\left| m^2\left((\sum_{j=1}^{d}\mu_j p_j)^2+1\right) - \left(\sum_{j=1}^{d+1}\mu_j\hat{p}_j\right)^2\right| > 0.$$

Then for such $(p_1,\cdots,p_d)\in\mathbb{Z}^d$*, L is torus equivariantly big, that is*

$$\dim_{\mathbb{C}} H^0_{mp_1,\cdots,mp_d}(M,L^m)=O(m^n).$$

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